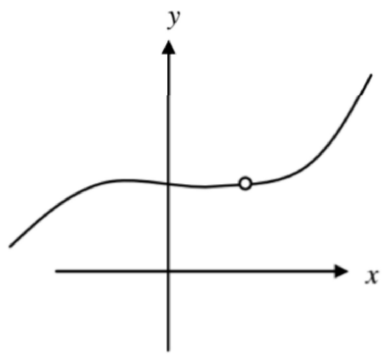
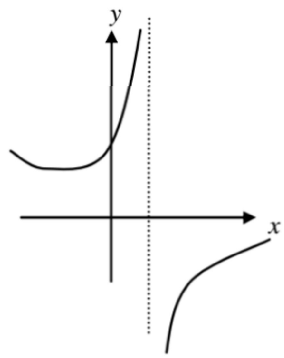
AB Calculus: Continuity

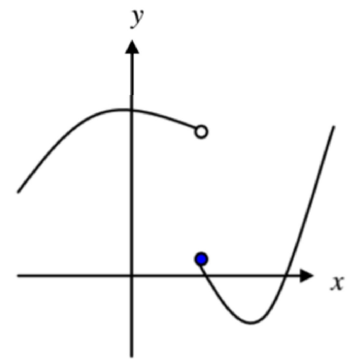
A function is continuous if you can draw the function without ever lifting your pencil. The following graphs demonstrate three types of discontinuous graphs.



Removable Discontinuity
Hole in the graph



Non-Removable Discontinuity
Vertical Asymptote



Non-Removable Discontinuity
Jump

There are two types of discontinuities, removable and non-removable. A hole in the graph is an example of a removable discontinuity. It is considered removable because you can easily make the graph continuous again by filling the hole. Vertical asymptotes and jumps are examples of non-removable discontinuities. They cannot be made continuous without drastically changing the function itself.

Example 1: Find the points (intervals) at which the function below is continuous, and the points at which it is discontinuous over the interval $0 < x < 5$.

continuous

$0 < x < 1$

$1 < x < 2$

$2 < x < 4$

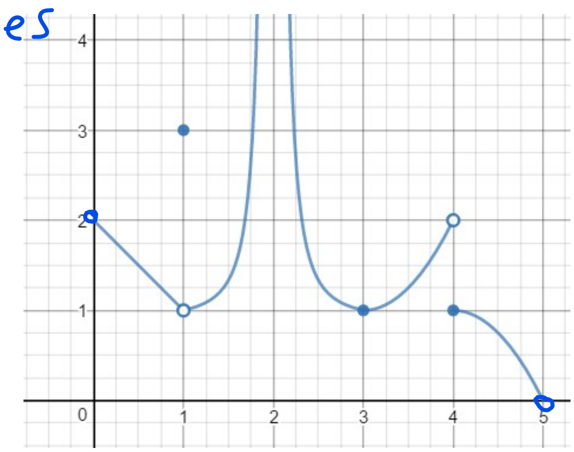
$4 \le x < 5$

Discontinuities @ $x =$

$1 \rightarrow$ Hole

$2 \rightarrow$ V.A

$4 \rightarrow$ Jump



Continuity at a point

A function $y = f(x)$ is continous at point c if

→ definition of continuity

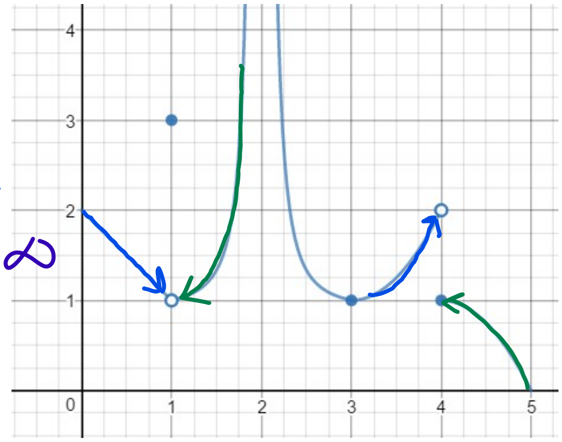
$$\boxed{\lim_{x \to c} f(x) = f(c)}$$

Or in other words,

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c)$$

"The limit from the left, equals the limit from the right, equals the function value."

Example 2: For $c = 1$, 2, and 4, find $f(c)$, $\lim_{x\to c^+} f(x)$, $\lim_{x\to c^-} f(x)$, and $\lim_{x\to c} f(x)$. Take notice how each part of the definition of continuity is important.



$c=1$

$\lim_{x\to 1^-} f(x) = 1$ $\lim_{x\to 1^+} f(x) = 1$ $\lim_{x\to 1} f(x) = 1$

$f(1) = 3$

$c=2$

$\lim_{x\to 2^-} f(x) \to \infty$ $\lim_{x\to 2^+} f(x) \to \infty$ $\lim_{x\to 2} f(x) \to \infty$

$f(2)$ undefined

$c=4$ $\lim_{x\to 4^-} f(x) = 2$ $\lim_{x\to 4^+} f(x) = 1$

$\lim_{x\to 4} f(x)$ Does not Exist $f(4) = 1$

You don't always get a picture, so you will have to be able to do this algebraically as well.

Example 3: Determine whether each function is continuous or not. If it is not continuous, use the definition of continuity to explain why.

a) $f(x) = \dfrac{1}{x-1}$ NO

$\hookrightarrow x=1$ is the VA

$\lim_{x\to 1} f(x)$ Does not exist

there $f(x)$ is not cont at $x=1$

NO b) $g(x) = \dfrac{2x^2+x-6}{x+2} \rightarrow \dfrac{(2x-3)\cancel{(x+2)}}{\cancel{(x+2)}}$

Hole at $x=-2$

$\lim_{x\to -2} f(x) = 2(-2)-3 = -7$

$f(-2)$ undefined

c) $h(x) = \begin{cases} -2x+3, & x<1 \\ x^2, & x \geq 1 \end{cases}$

$\lim_{x\to 1^-} h(x) = -2(1)+3 = 1$ $\lim_{x\to 1^+} h(x) = 1^2 = 1$ Therefore $\lim_{x\to 1} h(x) = 1$

Now, check for continuity

$h(1) = 1^2 = 1$

So the $h(x)$ function IS continuous

Example 4: Use the definition of continuity to find the value of a so that g(x) will be continuous for all real numbers.

a) $g(x) = \begin{cases} x^2+7, & x \geq 1 \\ x+a, & x<1 \end{cases}$ → Just plug this in for "x"

$1^2+7 = 1+a$

$8 = 1+a$

$a = 7$

b) $h(x) = \begin{cases} \dfrac{x^4-1}{x-1}, & x \neq 1 \\ k, & x = 1 \end{cases}$

$\dfrac{1^4-1}{1-1}$ IND!

$\dfrac{x^4-1}{x-1} = \dfrac{(x^2+1)(x^2-1)}{x-1} = \dfrac{(x^2+1)(x+1)\cancel{(x-1)}}{\cancel{x-1}}$

$(1^2+1)(1+1) = k$

$k=4$

Properties of Continuity

If b is a real number and f and g are continuous at $x = c$, then the following functions are continuous at c.

1. Constant multiple: $b \cdot f$
2. Sum and Difference: $f \pm g$
3. Product: $f \cdot g$
4. Quotient: $\frac{f}{g}$; $g(c) \neq 0$

Extended Functions

An extended function is a function that is obtained after a discontinuity is removed. To write an extended function, find where the hole in the graph, then write a piecewise function to fill the hole.

Example 4: Write an extended function to remove the removable discontinuity from the function $f(x)$.

a) $f(x) = \dfrac{x^2 - 6x + 5}{x - 1}$

$$\frac{(x-5)(x-1)}{x-1}$$

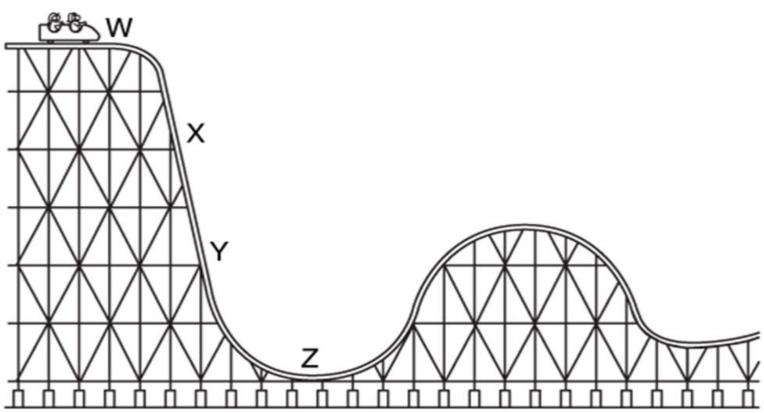
$$f(x) = \begin{cases} x-5 & , x \neq 1 \\ -4 & , x = 1 \end{cases}$$

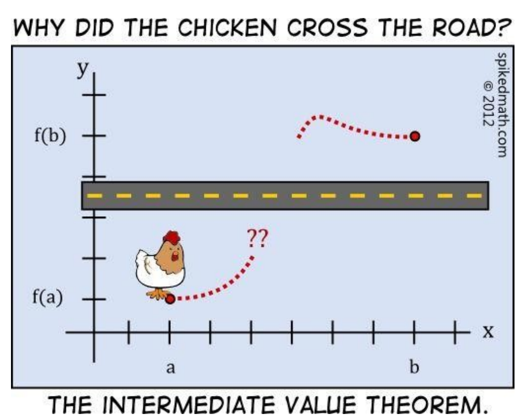
b) $f(x) = \dfrac{x^2 - 3x - 18}{x + 3}$

$$\frac{(x-6)(x+3)}{x+3}$$

$$f(x) = \begin{cases} x-6 & , x \neq -3 \\ -9 & , x = -3 \end{cases}$$

Intermediate Value Theorem (IVT)





For the car to go from point W to point Z safely, do you have to go through points X and Y? Why or Why not?

Yes Stay on track

Intermediate Value Theorem (IVT)

these are x-values

these are y-values

If f is continuous on the closed interval $[a, b]$ then f takes every value between $\boxed{f(a) \text{ and } f(b)}$.

Suppose k is a value between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that $f(c) = k$.

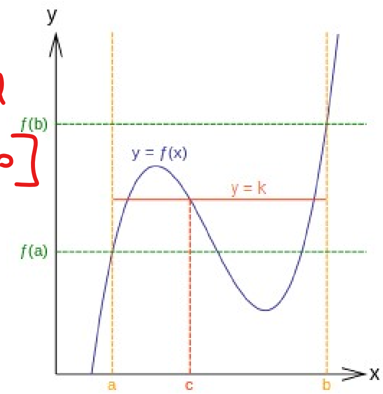
is a y-value

x-value

The Intermediate Value Theorem tells you that at least one c exists, but it does not give you a method of finding c. This theorem is an example of an **existence theorem**.

$\in$ → is an element of

Example 5: In the Intermediate Value Theorem …



a) What are the necessary requirements in order to apply this theorem?

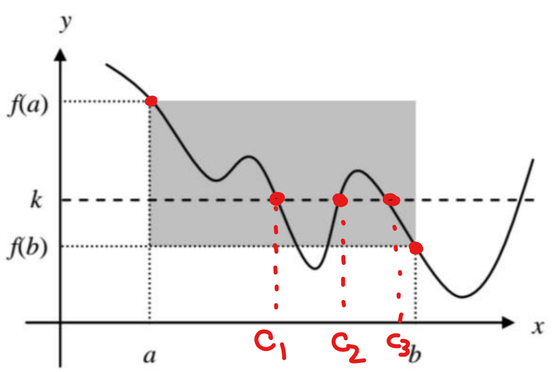
1 f has to continuous on closed interval

(in notation form $x \in [a,b]$

b) k is on which axis? y

c) c is on which axis? x

Example 6: Consider the function below and answer the questions.



a) Is f continuous on $[a, b]$? yes

b) Is k between $f(a)$ and $f(b)$? yes

c) In this example, if $a < c < b$, then there are <u>3</u> $c's$ such that $f(c) = k$.

d) Label the $c's$ on the graph as $c_1, c_2, \ldots$

Example 7: Verify that the Intermediate Value Theorem applies to the following function $f(x)$ over the interval $\left[\frac{5}{2}, 4\right]$, explain why IVT guarantees an x-value of c where $f(c) = 6$, and find c.

$$f(x) = \frac{x^2 + x}{x - 1}$$

f(x) is not continuous if x=1, but 1 is not in $[\frac{5}{2},4]$

therefore f(x) is cont on $[\frac{5}{2},4]$

$$f\left(\tfrac{5}{2}\right) = \frac{\left(\frac{5}{2}\right)^2 + \frac{5}{2}}{\frac{5}{2} - 1\left(\frac{2}{2}\right)} = \frac{\frac{25}{4} + \frac{10}{4}}{\frac{3}{2} \to \frac{6}{4}} \to \frac{\frac{35}{4}}{\frac{6}{4}} \to \frac{35}{6} = 5\tfrac{5}{6}$$

$$f(4) = \frac{4^2 + 4}{4 - 1} = \frac{16 + 4}{3} = \frac{20}{3} = 6\tfrac{2}{3}$$

By IVT, since $5\frac{5}{6} < 6 < 6\frac{2}{3}$ there is a $\frac{5}{2} < c < 4$ such that $f(c) = 6$

Find c

$$6 = \frac{x^2 + x}{x - 1} \to 6(x-1) = x^2 + x \to 6x - 6 = x^2 + x$$

$$\underline{-6x + 6 \qquad -6x + 6}$$

$$0 = x^2 - 5x + 6$$

$$0 = (x-2)(x-3)$$

outside $[\frac{5}{2},4]$

Algebra says $x = \cancel{2}, 3$

SO c = 3