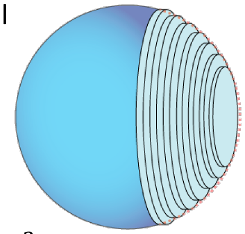
AB Calculus Volume of Solids with Known Cross Sections **Name: ______________________________**

[Geogebra](https://goo.gl/0Hk15y) Demo to go with this notesheet: https://goo.gl/0Hk15y



Today, we are going to expand upon the basic ideas behind the disk/washer methods to deal with volumes of solids with other known cross sections. What is a cross section? A cross section of a three dimensional figure is the intersection of a plane and that figure. It would be like cutting an object and then looking at the face of where you just cut. When using the disk method, the cross sections of the solid are circles, like in the picture to the right. The cross sections were taken perpendicular to the axis of revolution. To find the volume, we found the area of the circle by substituting the radius into the area of a circle formula, $A = \pi r^2$, multiplying by dx or dy to get the volume of one slice, then integrating to find the volume of the solid. This same process can be generalized to find the volume of any 3-D shape with cross sections taken perpendicular to an axis whose area can be modeled by a single-variable function.

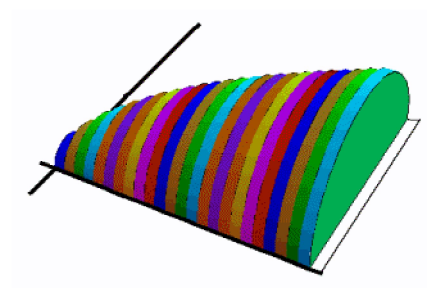
Definition of Volume

Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane through x and perpendicular to the x-axis is $A(x)$, where A is a continuous function, then the volume of S is

$$V = \int_a^b A(x)dx$$

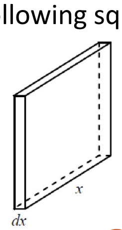
Let S be a solid that lies between $y = c$ and $y = d$. If the cross-sectional area of S in the plane through y and perpendicular to the y-axis is $A(y)$, where A is a continuous function, then the volume of S is

$$V = \int_c^d A(y)dy$$



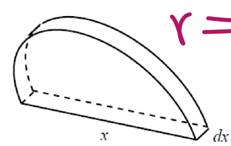
Here is the basic idea. You will be given a region defined by a number of functions. We will graph that region on the x and y axes. Then, we will lay the region flat and build upon that region a solid which has the same cross section no matter where you slice it. The picture to the right is an example of one of these solids. No matter where you slice, the cross section would be a semicircle. To find the volume, we find the area of the cross section, multiply by dx or dy to get the volume of one slice, then integrate over the interval to get the volume of the solid. So, the question becomes, how do you find the area expression? Let's look at a few examples.

Example 1 Write an expression for the volume of the following square slice.



$x^2 dx$

Example 2 Write an expression for the volume of the following semicircular slice.



$r = \frac{x}{2}$

$A_{circle} = \pi r^2$

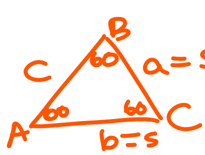
$A_{circle} = \pi\left(\frac{x}{2}\right)^2 = \frac{\pi x^2}{4}$

$A_{semi\text{-}circle} = \frac{1}{2} \cdot \frac{\pi x^2}{4}$

$A_{sc} = \frac{\pi x^2}{8}$

$\boxed{V = \frac{\pi x^2}{8} dx}$

$A = \frac{1}{2} ab \sin C$



$\frac{1}{2} s^2 \sin 60° = \frac{1}{2} s^2 \frac{\sqrt{3}}{2} = \frac{s^2\sqrt{3}}{4}$

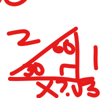
What about other shapes? Here are a few more geometric formulas of some shapes that show up a lot.

Triangles: $A = \frac{1}{2}bh$

Rectangle: $A = lw$

Equilateral Triangles: $A = \frac{\sqrt{3}}{4}s^2$

$30° \to 1$ $\quad 60° \to \sqrt{3}$ $\quad 90° \to 2$



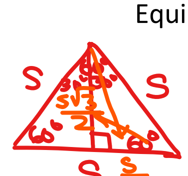
$\sin 30° = \frac{1}{2}$

$x^2 + 1^2 = 2^2$

$x^2 + 1 = 4$

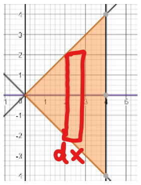
$x^2 = 3$

$x = \sqrt{3}$



$\frac{1}{2} s \cdot \frac{s\sqrt{3}}{2} = \frac{1 s^2 \sqrt{3}}{4}$

Example 3 Find the volume of a solid whose base is bounded by the graphs of the functions $y = x$, $y = -x$, and $x = 4$ whose cross sections perpendicular to the x-axis are squares.

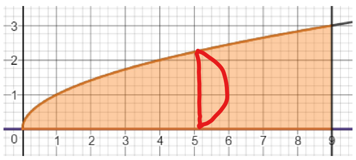


$$V=\int_0^4 (2x)^2\,dx = \int_0^4 4x^2\,dx = \frac{4}{3}x^3\Big|_0^4 = \frac{4}{3}(4)^3 - 0 = \frac{256}{3}$$

Area of square = side $(s)^2$

base of cross-section is just top - bottom $(x - -x) = 2x$

Example 4 Find the volume of a solid whose base is bounded by the graphs of the functions $y = \sqrt{x}$, $x = 9$, and the x-axis whose cross sections perpendicular to the x-axis are semicircles.

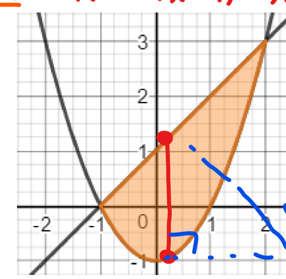


Top - bottom $= \sqrt{x} - 0 = \sqrt{x} =$ diameter

so radius $= \frac{\sqrt{x}}{2}$, Area semicircle $= \frac{1}{2}\pi\left(\frac{\sqrt{x}}{2}\right)^2$

$$V=\int_0^9 \frac{1}{2}\pi\left(\frac{\sqrt{x}}{2}\right)^2 dx \rightarrow \frac{1}{2}\pi\,\frac{1}{4}\int_0^9 x\,dx \rightarrow \frac{\pi}{8}\left[\frac{x^2}{2}\right]_0^9 = \boxed{\frac{81\pi}{16} - 0}$$

Example 5 Find the volume of the solid whose base is bounded by the graphs of $y = x + 1$ and $y = x^2 - 1$, with the following cross sections taken perpendicular to the x-axis. Length of base $= x+1-(x^2-1) = x+1-x^2+1$



a) Rectangles whose height is twice the base

$A = L \times W = H \times B = \underbrace{2(-x^2+x+2)}_{H=2B}\underbrace{(-x^2+x+2)}_{B}$ $\quad V=\int_{-1}^{2} 2(-x^2+x+2)^2\,dx$

$= 16\ 2$

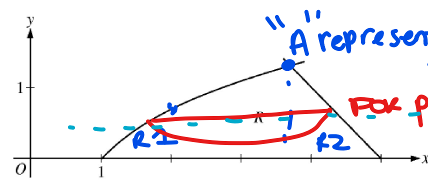
b) Equilateral triangles

$A = \frac{\sqrt{3}s^2}{4}$ $\quad V=\int_{-1}^{2}\frac{\sqrt{3}}{4}(-x^2+x+2)^2\,dx \approx 3\ 507$

c) Isosceles right triangles with a leg in the region.

$A = \frac{1}{2}bh = \frac{1}{2}b^2$ $\quad V=\int_{-1}^{2}\frac{1}{2}(-x^2+x+2)^2\,dx = 4\ 05$

Example 6 Let R be the region in the first quadrant bounded by the x-axis and the graphs of $y = \ln x$ and $y = 5 - x$, as shown in the figure.



a) Find the area of R. $A = 3\ 69$

$$\text{Area} = \int_1^A \ln x\,dx + \int_A^5 (5-x)\,dx \approx 2986$$

b) Region R is the base of a solid. For the solid, each cross section perpendicular to the x-axis is a square. Write, but do not evaluate, an expression involving integrals that gives the volume of the solid.

A square in section 1 $= (\ln x)^2$ $\quad$ A square in $R_2 = (5-x)^2$ $\quad V=\int_1^A(\ln x)^2\,dx+\int_A^5(5-x)^2\,dx$

c) Region R is the base of a solid. For the solid, each cross section perpendicular to the y-axis is a semicircle. Write, but do not evaluate, an expression involving integrals that gives the volume of the solid.

$y = \ln x \rightarrow x = e^y$ $\quad$ diameter $= 5 - y - e^y$

$y = 5 - x \rightarrow x = 5 - y$ $\quad$ radius $= \frac{5-y-e^y}{2}$

$$V=\int_0^{1.306558641}\frac{1}{2}\pi\left(\frac{5-y-e^y}{2}\right)^2 dy$$

d) The horizontal line $y = k$ divides R into two regions of equal area. Write, but do not solve, an equation involving one or more integrals whose solution gives the value of k.

$$\int_0^K (5-y-e^y)\,dy = \int_K^{1.306558641}(5-y-e^y)\,dy$$