

$$0 \downarrow, 1 \downarrow, 2 \downarrow, 3 \downarrow, 4 \rightarrow 10, 11, 10, 7, 2 \quad \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}$$

AB Calculus Definite Integrals and Antiderivatives Notesheet Name: \_\_\_\_\_

$$1 = \frac{4-0}{4}$$

1. Approximate the area under  $y = -x^2 + 2x + 10$  over  $[0, 4]$  using 4 subintervals and the given method.

a) LRAM  $1(10+11+10+7) = 38$

b) RRAM  $1(11+10+7+2) = 30$

c) MRAM  $1(10.75+10.75+8.75+4.75) = 35$

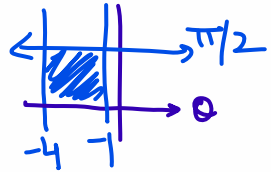
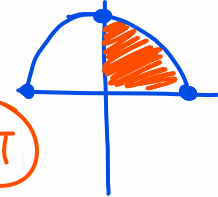
d) Trapezoid Rule  $\frac{1}{2}(10+2(11+10+7)+2)(1) = 34$

2. Evaluate the following integrals.

a)  $\int_{-4}^{-1} \frac{\pi}{2} d\theta = (3) \left( \frac{\pi}{2} \right) = \frac{3\pi}{2} \checkmark$   
 $\frac{\pi}{2} \theta \Big|_{-4}^{-1} = \frac{\pi}{2} \theta - \frac{4\pi}{2} = \frac{3\pi}{2} \checkmark$

b)  $\int_0^4 \sqrt{16-x^2} dx$

$$\frac{\pi(4)^2}{4} = 4\pi$$



4. Let  $f$  be the function given by  $f(x) = x^3 - 7x + 6$ . Find the number  $c$  that satisfies the conclusion of the Mean Value Theorem of Derivatives for  $f$  on the closed interval  $[1, 3]$ .

$f(3) = 12$   
 $f(1) = 0$

$\frac{12-0}{3-1} = \frac{12}{2} = 6$

$f'(x) = 3x^2 - 7$

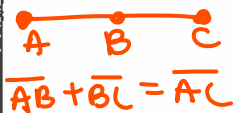
$6 = 3x^2 - 7$   
 $13 = 3x^2$   
 $\frac{13}{3} = x^2$

$x = \sqrt{\frac{13}{3}}$

MVT  
 $\frac{f(b)-f(a)}{b-a} = f'(c)$

In this last section we defined the definite integral as a limit of a Riemann Sum, thus we can use the properties of limits to develop properties of the definite integral. The proofs of each of the rules below are derived directly from the properties of limits and Riemann Sums.

Rules for Definite Integrals			
#	Rule	Notation	Statement
1.	Order of Integration	$\int_a^b f(x) dx = - \int_b^a f(x) dx$	If you reverse the order of integration, you get the opposite answer.
2.	Zero	$\int_a^a f(x) dx = 0$	This should make sense if you think about the area of a rectangle with no width.
3.	Constant Multiple	$\int_a^b k \cdot f(x) dx = k \cdot \int_a^b f(x) dx$	Taking the constant out of the integral many times makes it simpler to integrate.
4.	Sum and Difference	$\int_a^b [f(x) \pm g(x)] dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$	This allows you to integrate functions that are added or subtracted separately.
5.	Additivity	$\int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx$	Pay close attention to the limits of integration ... this comes in handy when dealing with total area or other functions where we need to break them into smaller parts.



Note: Notice how there are no rules here for two functions that are multiplied or divided ... that comes later!

Example 1: Given  $\int_2^6 f(x)dx = 10$  and  $\int_2^6 g(x)dx = -2$ , find the following:

a)  $\int_2^6 [f(x) + g(x)]dx = \int_2^6 f(x)dx + \int_2^6 g(x)dx = 10 + (-2) = \boxed{8}$

b)  $\int_2^6 [g(x) - f(x)]dx = \int_2^6 g(x)dx - \int_2^6 f(x)dx = -2 - 10 = \boxed{-12}$

c)  $\int_2^6 3f(x)dx = 3 \int_2^6 f(x)dx = 3(10) = \boxed{30}$

d)  $\int_2^6 [f(x) + 2]dx = \int_2^6 f(x)dx + \int_2^6 2dx = 10 + 8 = \boxed{18}$

Example 2: Given  $\int_0^5 f(x)dx = 10$  and  $\int_5^7 f(x)dx = 3$ , find the following:

a)  $\int_0^7 f(x)dx = \int_0^5 f(x)dx + \int_5^7 f(x)dx = 10 + 3 = \boxed{13}$

b)  $\int_5^0 f(x)dx = -10$

c)  $\int_5^5 f(x)dx = \boxed{0}$

d)  $\int_0^5 3f(x)dx = 3 \int_0^5 f(x)dx = 3(10) = 30$

1. (Calculator Permitted) If the midpoints of 4 equal-width rectangles is used to approximate the area enclosed between the x-axis and the graph of  $y = 4x - x^2$ , the approximation is  $4\left(\frac{5}{2}\right) - \left(\frac{5}{2}\right)^2 = 10 - \frac{25}{4} = 3.75$

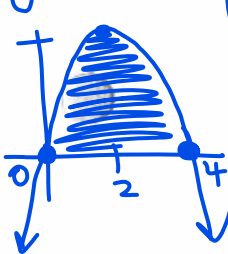
(A) 10 (B) 10.5 (C) 10.666 (D) 10.75 (E) 11

$\frac{4-0}{4} = 1$

$4\left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)^2 = 2 - \frac{1}{4} = 1.75$

$4\left(\frac{3}{2}\right) - \left(\frac{3}{2}\right)^2 = 6 - \frac{9}{4} = 3.75$

$1(1.75 + 3.75 + 3.75 + 1.75) = 11$



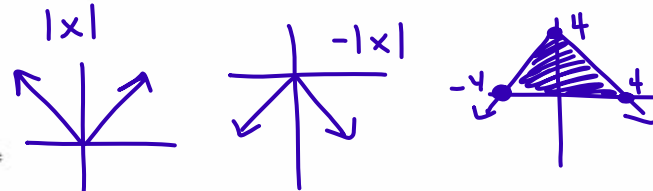
2. If  $\int_2^5 f(x)dx = 18$ , then  $\int_2^5 (f(x) + 4)dx = \int_2^5 f(x)dx + \int_2^5 4dx = 18 + 12 = 30$

(A) 20 (B) 22 (C) 23 (D) 25 (E) 30

OR

3.  $\int_{-4}^4 (4 - |x|)dx = \int_{-4}^4 4dx - \int_{-4}^4 |x|dx = 4(8) - 2(8) = 16$

(A) 0 (B) 4 (C) 8 (D) 16 (E) 32



4. If  $\int_a^b f(x)dx = a + 2b$ , then  $\int_a^b (f(x) + 3)dx = \int_a^b f(x)dx + \int_a^b 3dx = a + 2b + 3b - 3a = 5b - 2a$

(A)  $a + 2b + 3$  (B)  $3b - 3a$  (C)  $4a - b$  (D)  $5b - 2a$  (E)  $5b - 3a$

### The Average Value of a Function

If  $f$  is integrable on  $[a, b]$ , its average value on  $[a, b]$  is given by

$$\text{Average Value} = \frac{1}{b-a} \int_a^b f(x) dx \text{ or } \text{Average Value} = \frac{\int_a^b f(x) dx}{b-a}$$

The average value of a function is just the integral over the interval.

To get a more geometric idea of what the average value is, complete the following example.

#### Example 7

- Graph the function  $y = x^2$  on  $[0, 3]$  on the grid to the right.
- Set up a definite integral to find the average value of  $y$  on  $[0, 3]$ , then use your calculator to evaluate the definite integral.

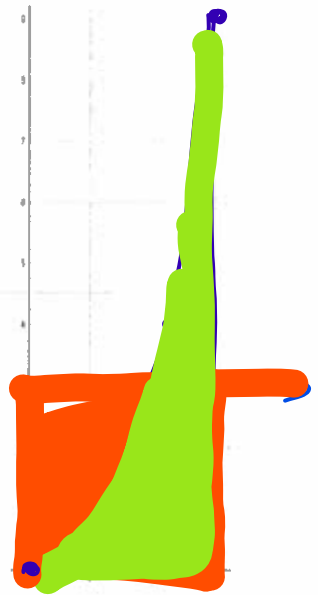
$$\frac{\int_0^3 x^2 dx}{3-0} \rightarrow \frac{\frac{1}{3}x^3 \Big|_0^3}{3-0} = \frac{\frac{1}{3}(3)^3 - 0}{3} = \textcircled{3}$$

- Graph this value as a function on the grid to the right. Does this function ever actually equal this value? If so, at what point(s) in the interval does the function assume its average value?

$$x^2 = 3 \rightarrow \text{so } x = \sqrt{3} \approx 1.73 \dots$$

- What do you suppose is the relationship between the area the  $x$ -axis and the curve  $y = x^2$  on  $[0, 3]$  and the area of the rectangle formed using the average value as the height and the interval  $[0, 3]$  as the width?

Same they are



### The Mean Value Theorem for Definite Integrals

The Mean Value Theorem for Integrals basically says that if you are finding the area under a curve between  $x = a$  and  $x = b$ , then there is some number  $c$  between  $a$  and  $b$  whose function value you can use to form a rectangle that has an area equal to the area under the curve.

#### Example 8

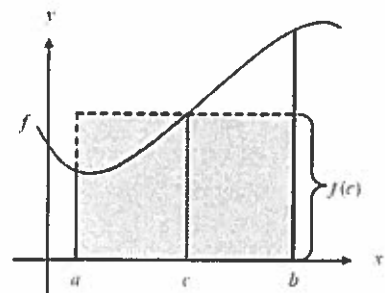
- What is an expression that could be used to determine the area under the curve from  $a$  to  $b$ ?

$$\int_a^b f(x) dx$$

- What is the area of the shaded rectangle?

$$f(c) \cdot (b-a)$$

This value of  $f(c)$  is just the average value of  $f$  on the interval  $[a, b]$ .



$$\frac{\int_a^b f(x) dx}{b-a} = f(c) \frac{(b-a)}{b-a}$$

So another way to look at this is the Mean Value Theorem for integrals just says that at some point within the interval the function MUST equal its average value.

### Mean Value Theorem for Integrals

If  $f$  is continuous on  $[a, b]$ , then at some point  $c$  in  $[a, b]$

$$f(c) = \frac{1}{b-a} \int_a^b f(x) dx$$

Once again, this is an existence theorem. It tells us that  $c$  exists, but does not help you find it.

It is greatly important that you understand the difference between average rate of change and average value. For now, understand that the average rate of change is the slope between two points on a given function and the average value is the integral divided by the interval.

Example 9 Find the average value of the function  $y = -3x^2 - 1$  over the interval  $[0, 1]$ .

$$\frac{\int_0^1 (-3x^2 - 1) dx}{1-0} = \frac{-x^3 - x \Big|_0^1}{1-0} = \frac{(-1-1) - (-0-0)}{1} = \frac{-2}{1} = -2$$

Example 10: (2014 AP Calculus AB Free Response Question #1)

Grass clippings are placed in a bin, where they decompose. For  $0 \leq t \leq 30$ , the amount of grass clippings remaining in the bin is modeled by  $A(t) = 6.687(0.931)^t$ , where  $A(t)$  is measured in pounds and  $t$  is measured in days.

a) Find the average rate of change of  $A(t)$  over the interval  $0 \leq t \leq 30$ . Indicate units of measure.

$$\frac{A(30) - A(0)}{30 - 0} = -197 \text{ pounds/day}$$

b) Find the value of  $A'(15)$ . Using correct units, interpret the meaning of the value in the context of the problem.

$$A'(15) = -164 \text{ pounds/day}$$

the amount of grass clippings is decreasing at a rate of 164 lbs/day on the 15th day

c) Find the time  $t$  for which the amount grass clippings in the bin is equal to the average amount of grass clippings in the bin over the interval  $0 \leq t \leq 30$ .

$$A(t) = \frac{\int_0^{30} A(t) dt}{30-0} \quad t = 12.415$$

d) For  $t > 30$ ,  $L(t)$ , the linear approximation to  $A$  at  $t = 30$ , is a better model for the amount of grass clippings remaining in the bin. Use  $L(t)$  to predict the time at which there will be 0.5 pounds of grass clippings remaining in the bin. Show the work that leads to your answer.

$$y - y_1 = m(x - x_1)$$

$$y = m(x - x_1) + y_1$$

$$L(t) = (A'(30))(t - 30) + A(30)$$

$$.5 = (A'(30))(t - 30) + A(30)$$

$$t = 35.054$$