AB Calculus: The Derivative of a Function **Name:**____________________________________

The derivative of a function is one of the two main concepts in calculus. The other is called an integral, which we will not get until later, and both are really just limits when you look under the hood. The only change from the limit definition we used before, is that we are going to treat the derivative as a function derived from f.

The Limit Definition of Derivative

The **derivative** of a function f with respect to the variable x is the function f' whose value at x is

f prime of x ←

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h},$$

you can take the derivative

provided the limit exists.

Anywhere that the derivative exists, we say that the function $\boxed{\text{is } \textit{differentiable}}$.

Thus, the derivative f' is a function that gives the slope of the function f at any point.

Other Notation used with derivatives (we will use most of these so remember them).

if equation is $y =$

the derivative will be $\frac{dy}{dx}$ or y'

Example 1: Use the definition of derivative to find $f'(x)$. $= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

a) $f(x) = 3x + 2$

$f(x+h) = 3(x+h) + 2 = 3x + 3h + 2$

$\lim_{h\to 0} \frac{3x+3h+2-3x-2}{h} \to \lim_{h\to 0} \frac{3h}{h} = \boxed{3}$ so, $\boxed{f'(x) = 3}$

b) $f(x) = x^3 + x^2$

$f(x+h) = (x+h)^3 + (x+h)^2 = x^3 + 3x^2h + 3xh^2 + h^3 + x^2 + 2xh + h^2$

$f(x) = x^3 + x^2$ $\quad - x^3 \quad -x^2$

$\lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 + 2xh + h^2}{h}$

$\lim_{h\to 0} 3x^2 + 3xh + h^2 + 2x + h \Rightarrow \boxed{3x^2 + 2x = f'(x)}$

Modified Form of the Limit Definition of the Derivative

The **numeric value** of the derivative of a function f at the point $(c, f(c))$ is given as

$$f'(c) = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h},$$

provided the limit exists.

If you are asked to find the derivative at a point, you have two options: you can find the formula first then plug in the point, or you can plug in the point in the beginning. However, if you need to find the derivative at multiple points, I suggest finding the formula first to save yourself some time.

Example 2: If $f(x) = 5x^2 + 2x$, find $f'(2)$.

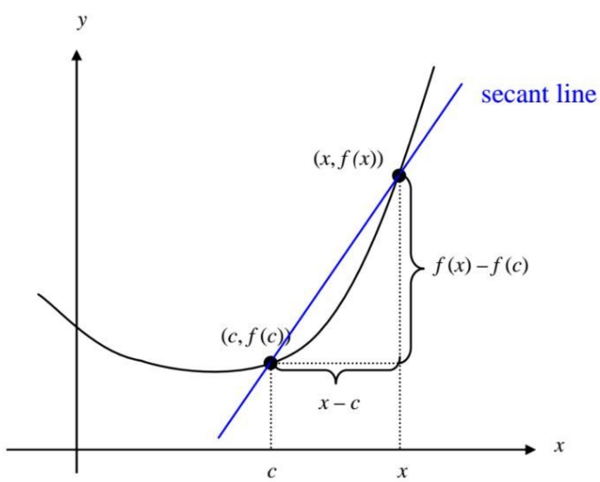
$$f(x+h) = 5(x+h)^2 + 2(x+h)$$
$$= 5(x^2+2xh+h^2) + 2x+2h$$
$$= 5x^2+10xh+5h^2+2x+2h$$
$$\lim_{h\to 0} \frac{5x^2+10xh+5h^2+2x+2h-5x^2-2x}{h}$$
$$\lim_{h\to 0} \frac{10xh+5h^2+2h}{h}$$
$$\lim_{h\to 0} 10x+5h+2$$
$$\boxed{f'(x) = 10x+2}$$
$$\boxed{f'(2) = 10(2)+2 = 22}$$

Alternative Definition of Derivative

An alternative definition of the derivative of f at point c is

$$f'(c) = \lim_{x\to c} \frac{f(x) - f(c)}{x - c},$$

provided the limit exists.



What this alternative definition allows us to do is to examine the behavior of a function as x approaches c from the left or the right. The limit exists (and thus the derivative) as long as the left and right limits exist and are equal.

Example 3: Use the alternative definition of the derivative to find the derivative of $f(x) = \frac{1}{\sqrt{x}}$ at $x = 4$

$$\lim_{x\to 4} \frac{f(x)-f(4)}{x-4}$$

$$f(4) = \frac{1}{\sqrt{4}} = \frac{1}{2}$$

$$\lim_{x\to 4} \frac{\frac{2}{2}\cdot\frac{1}{\sqrt{x}} - \frac{1}{2}\cdot\frac{\sqrt{x}}{\sqrt{x}}}{x-4}$$

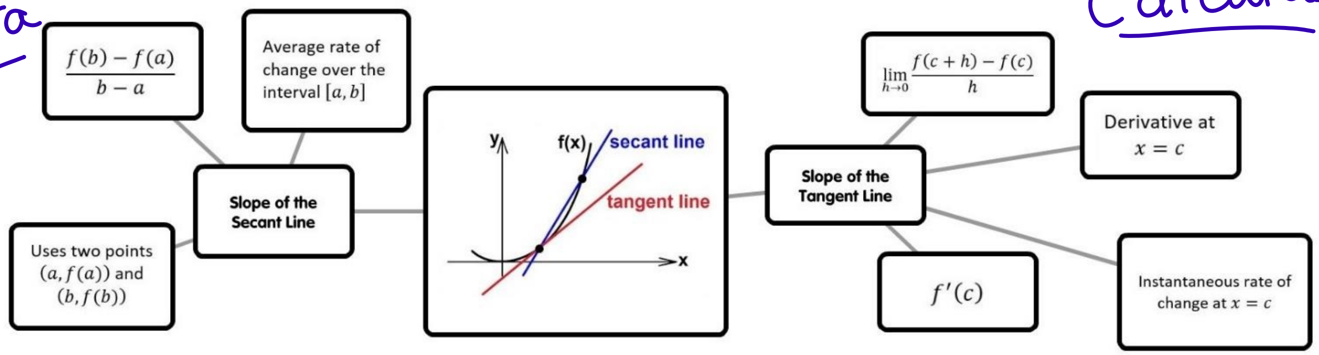
$$\lim_{x\to 4} \frac{\frac{2-\sqrt{x}}{2\sqrt{x}}}{x-4}\cdot\frac{2+\sqrt{x}}{2+\sqrt{x}}$$

$$\lim_{x\to 4} \frac{\cancel{4-x}\;-1}{2\sqrt{x}\,(2+\sqrt{x})\cancel{(x-4)}}$$

$$\frac{-1}{2\sqrt{4}\,(2+\sqrt{4})} = \frac{-1}{2(2)\cdot 4} = \boxed{\frac{-1}{16} = f'(4)}$$

It is important to formally lay out the meanings and distinctions between secant lines and tangent lines.

Algebra   Calculus



Example 4: If $f(x) = x^2 - 4x$, answer the following.

a) Find the average rate of change of $f(x)$ over the interval $[-1, 2]$.

x-values

$f(-1) = (-1)^2 - 4(-1) = 1+4 = 5$

$f(2) = (2)^2 - 4(2) = 4-8 = -4$

$\frac{f(2)-f(-1)}{2--1} = \frac{-4-5}{2+1} = \frac{-9}{3} = \boxed{-3}$

b) Find the instantaneous rate of change of $f(x)$ at the point where $x = 2$.

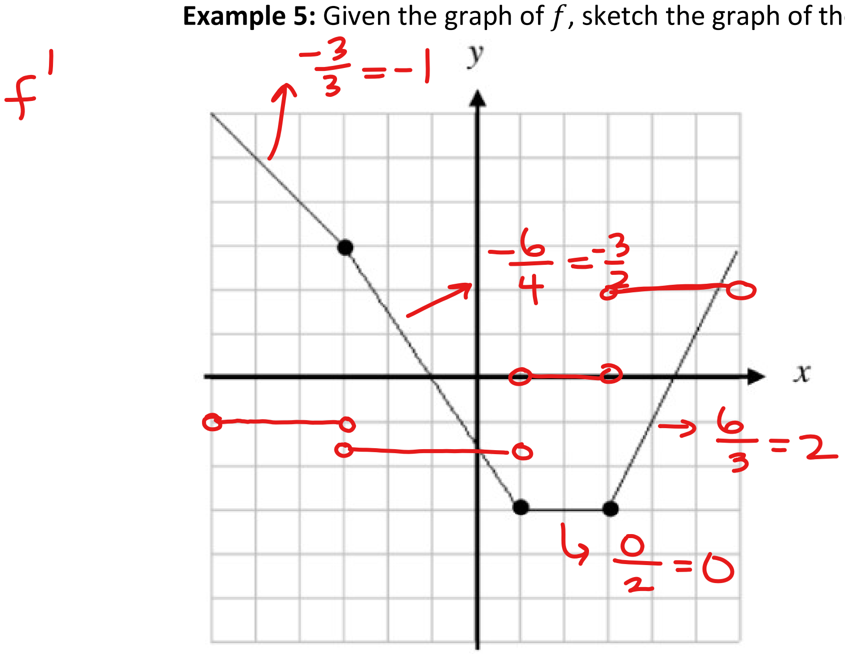
$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ or $f'(2) = \lim_{x\to 2}\frac{f(x)-f(2)}{x-2}$

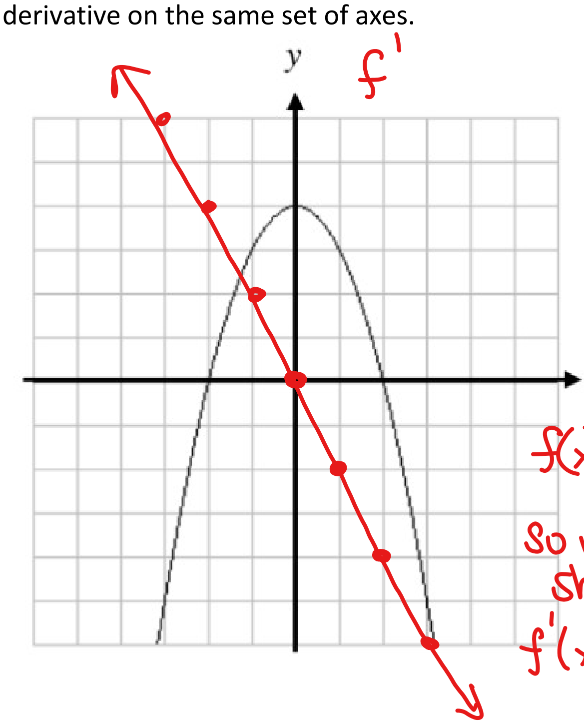
$f'(x) = 2x-4$ so $f'(2) = 2(2)-4 = \boxed{0}$

Relationships between the graphs of f and f'

Since a derivative at any point is equivalent to the slope of the function at that point, we can graph the derivative given a function and estimate what the original function looks like when we are given the graph of the derivative.

Example 5: Given the graph of f, sketch the graph of the derivative on the same set of axes.





$f(x) = -x^2 + 4$

So using shortcut $f'(x) = -2x$

initial condition (point on the graph)

Example 6: Given the graph of f' and $\boxed{f(0) = 2}$, sketch the graph of the function f on the same set of axes.

continuous

$f'(x) = -1$ if $x < 1$

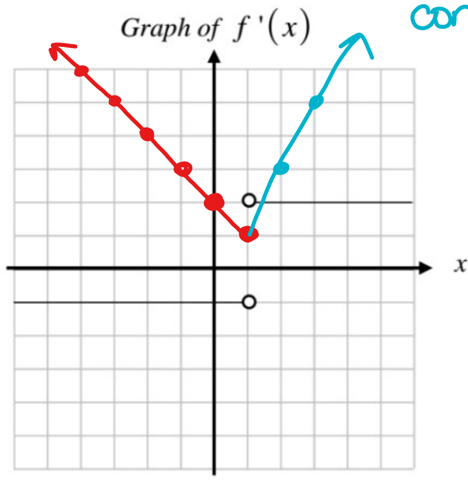
so $f(x) = -1x + c$

unknown

$f'(x) = 2$ if $x \geq 1$

$f(x) = 2x + c$

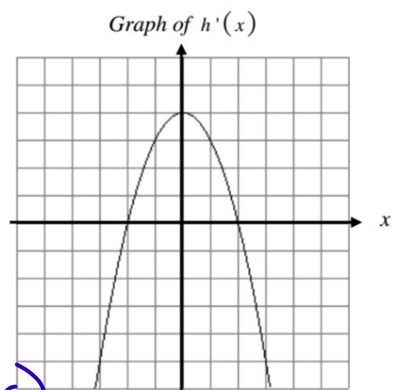
unknown



Why is it necessary to know $f(0) = 2$?

I need to "where" (so y-int is helpful)

Example 7: Suppose the graph below is the graph of the derivative of h.



a) What is the value of $h'(0)$? What does this tell us about $h(x)$?

$h'(0) = 4$

at $x = 0$, $h(x)$ is increasing at a rate of 4

b) Using the graph of $h'(x)$, how can we determine when the graph of $h(x)$ is going up? How about going down?

when $h'(x)$ is positive (above the x-axis)

when $h(x)$ is negative (below x-axis)

c) The graph of $h'(x)$ crosses the x-axis at $x = 2$ and $x = -2$. Describe the behavior of the graph of $h(x)$ at these points.

this is where the slope is 0, so the graph will change direction