AB Calculus: Rates of Change and Tangent Lines **Name: ______________________________**

The World Record Basketball Shot

A group called How Ridiculous became YouTube famous when they successfully made a basket from the top of Tasmania's Gordon Dam, 415 feet above the ground, breaking the World Record for the highest basketball shot on June 14, 2015. This record stood until a group called Dude Perfect made a basket from the top of a building at Texas Christian University (TCU) from a height of 533 feet, a World Record that still stands today. Use the information from the Dude Perfect World Record Shot to answer the following questions.

Part 1

a) If an object is dropped from an initial height, h_0, we can use the position function $s(t) = -16t^2 + h_0$ to model the height, s, in feet of an object that has fallen for t seconds. What is the position function for the basketball as the shot makes its way to the ground?

b) A moving body's average velocity during an interval of time is found by dividing the total change in position by the elapsed amount time. If the moving body's position is modeled by $s(t)$ and the interval of time is $[a, b]$, the average velocity can be found using the formula: $\frac{s(b) - s(a)}{b-a}$. Find the basketball's average velocity for the first 3 seconds of flight. Does this formula remind you of anything?

c) Find the basketball's average velocity between t = 2 and t = 3 seconds.

d) Find basketball's velocity at the instant t = 3 seconds.

e) What complication(s), if any, did you encounter when answering any of the previous questions?

Part 2

a) Find the average velocity between t = 2.5 and t = 3 seconds.

b) Find the average velocity between t = 2.9 and t = 3 seconds.

c) Find the average velocity between t = 2.99 and t = 3 seconds.

d) Find the average velocity between t = 2.999 and t = 3 seconds.

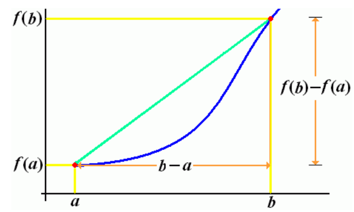
e) What is the basketball's velocity approaching at t = 3 seconds?

Secant Lines and Average Rate of Change

A **secant** line is a line joining two points on a function. Finding the slope of the secant line through the points $P_1(a, f(a))$ and $P_2(b, f(b))$ will tell you the average rate of change over the interval $[a, b]$. This was the process you were doing in the previous basketball shot scenario when you found the slope during intervals of time. The process to find the average rate of change is no different than the process you have used to find the slope of a line since Math 1, we will just write it a little differently.

Average Rate of Change

$$\frac{f(b) - f(a)}{b - a}$$

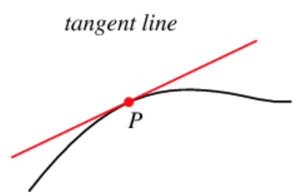


Example 1: Find the average rate of change of $f(x) = x^3 - x$ over the interval $[1, 3]$.

Tangent Lines and Instantaneous Rate of Change

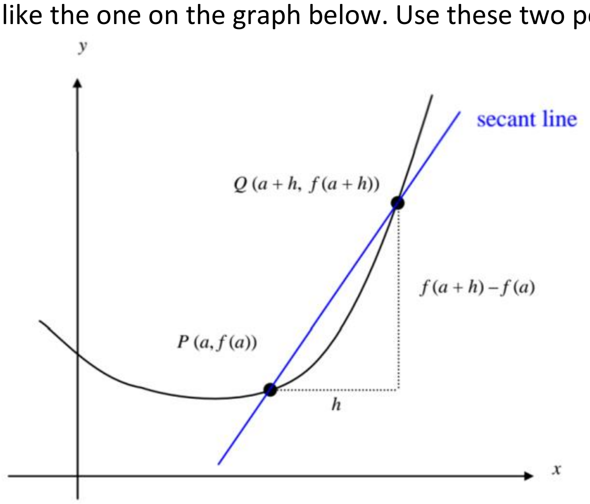
A **tangent** line is a line that touches a function at only one point. Finding the slope of the tangent line at a point will tell you the instantaneous rate of change at that point. Since we only have one point, using the average rate of change formula would cause a problem. If we try using the single point $P(a, f(a))$ in our average rate of change formula, we will get indeterminate form.

$$\frac{f(a) - f(a)}{a - a} = \frac{0}{0}$$



We can approximate the slope of the tangent line using a secant line. Just like in the basketball shot scenario, the closer the second point is to the point of tangency, the better the estimation for the slope of the tangent line at the point.

However, if we want to prove the slope of the curve is the value found in our approximation, we are going to have to find it algebraically. What if we add a very small amount (we will call this amount h), to our point $P(a, f(a))$ to create an arbitrary second point $Q(a + h, f(a + h))$? Then we would have a secant line scenario like the one on the graph below. Use these two points in the average rate of change formula to find the result.



If we substitute 0 for h, then we get indeterminate form again. However, if we find the limit as h approaches 0, then we can use algebraic techniques to evaluate the limit and find the slope of the tangent line at that single point, or the instantaneous rate of change!

Slope of a Curve at a Point

The slope of a line is always constant. The slope of a curve is constantly changing. Think of a curve as a roller coaster that you are riding. If for some reason the track were to just disappear, you would go flying off in the direction that you were traveling at that last instant before the track disappeared. The direction that you flew off to would be the slope of the curve at that point. To find this slope, we can find the limit as h approaches 0 of the result you got when finding the slope of the tangent line.

Slope of a Curve at a Point

The slope of the curve $y = f(x)$ at the point $P(a, f(a))$ is the number

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Provided the limit exists.

- The expression $\frac{f(a+h)-f(a)}{h}$ is called the **difference quotient**.
- Your goal will be to simplify the difference quotient, then evaluate the limit as h approaches 0.
- The difference quotient is simplified when you have divided out the h in the denominator in a correct way.

Example 2: Before we use this definition, be sure to be comfortable with the notation $f(a+h)$.

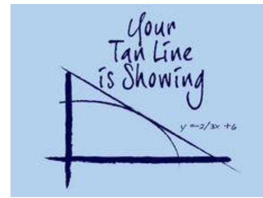
a) If $f(x) = \frac{1}{x}$, what is $f(a)$? $f(a+h)$?

b) If $f(x) = x^2 - 4x$, what is $f(a)$? $f(a+h)$?

c) If $f(x) = \sqrt{4x+1}$, what is $f(a)$? $f(a+h)$?

Example 3: Find the slope of the curve $f(x) = x^2$ at the point $(2, 4)$.

Finding the Equation of a Tangent Line to a Curve



If the slope of a curve at a point is the slope of the tangent line through the point, then we can find the equation of the line as well. Remember, to write the equation of a line, we need two things: a point on the line and the slope. You will generally be given the point, or at least one coordinate of it. We can find the slope by taking the limit as h approaches 0 of the difference quotient.

Example 4: Find the equation of the tangent line to the curve $f(x) = x^2$ at the point $(2, 4)$.

Recall that there are limits that fail to exist. Since the slope of a tangent line is a limit, the slope might not exist because the limit doesn't exist (discontinuity, vertical asymptote, oscillating function) or because the tangent line has a vertical slope.

For a tangent line that exists, but has an undefined slope, this definition does not quite fit. To cover the possibility of a vertical tangent line, we can use the following definition.

If f is continuous at a and

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \pm\infty$$

The vertical line $x = a$ is a vertical tangent line to the graph of f.

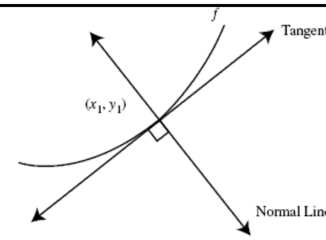
Example 5: What types of graphs would have vertical tangent lines?

Example 6: Back to the basketball shot example. Let $f(x) = -16x^2 + 533$. Find the slope of the curve at $x = 3$.

This would be the instantaneous velocity of the basketball at exactly 3 seconds!

Normal Lines

The **normal line** to a curve at a point is the line perpendicular to the tangent at that point.



To find the slope of the normal line, take the opposite reciprocal of the slope of the tangent line.

Example 7: Let $f(x) = \frac{1}{x+1}$.

a) Find the slope of the curve at $x = a$.

b) Find the slope of the curve at $x = 2$.

b) Write the equation of the tangent line to the curve at $x = 2$.

c) Write the equation of the normal line to the curve at $x = 2$.

AB Calculus: The Derivative of a Function **Name:**__________________________________

The derivative of a function is one of the two main concepts in calculus. The other is called an integral, which we will not get until later, and both are really just limits when you look under the hood. The only change from the limit definition we used before, is that we are going to treat the derivative as a function derived from f.

The Limit Definition of Derivative If function, f(x), then the derivative is f'(x) ↳ f prime of x

The **derivative** of a function f with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists.

Anywhere that the derivative exists, we say that the function is *differentiable*.

Thus, the derivative f' is a function that gives the slope of the function f at any point.

Other Notation used with derivatives (we will use most of these so remember them).

y' or $\frac{dy}{dx}$ → this notation implies that your equation is y in terms of x

Example 1: Use the definition of derivative to find $f'(x)$.

a) $f(x) = 3x + 2$ $f(x+h) = 3(x+h) + 2 = 3x+3h+2$

$$f'(x) = \lim_{h\to 0} \frac{\cancel{3x}+3\cancel{h}\cancel{+2}\cancel{-3x}\cancel{-2}}{\cancel{h}}$$

$= 3$

b) $f(x) = x^3 + x^2$ $f(x+h) = (x+h)^3 + (x+h)^2$

$$\lim_{h\to 0} \frac{\cancel{x^3}+3x^2\cancel{h}+3xh^{\cancel{2}}+h^{\cancel{3}2}\cancel{+x^2}+2x\cancel{h}+h^{\cancel{2}}\cancel{-x^3}\cancel{-x^2}}{\cancel{h}}$$

$$\lim_{h\to 0} 3x^2+3xh+h^2+2x+h = 3x^2+2x$$

Modified Form of the Limit Definition of the Derivative

The **numeric value** of the derivative of a function f at the point $(c, f(c))$ is given as

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h},$$

provided the limit exists.

If you are asked to find the derivative at a point, you have two options: you can find the formula first then plug in the point, or you can plug in the point in the beginning. However, if you need to find the derivative at multiple points, I suggest finding the formula first to save yourself some time.

Example 2: If $f(x) = 5x^2 + 2x$, find $f'(2)$.

$$\lim_{h\to 0} \frac{5(x+h)^2 + 2(x+h) - 5x^2 - 2x}{h}$$

$$\frac{\cancel{5x^2} + 10x\cancel{h} + 5h^{\cancel{2}} \cancel{+2x} + 2\cancel{h} - \cancel{5x^2} \cancel{-2x}}{\cancel{h}}$$

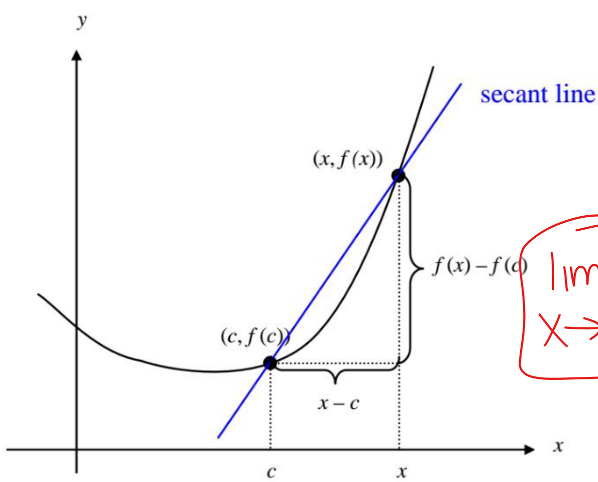
$$\lim_{h\to 0} 10x + 5h + 2 \Rightarrow f'(x) = 10x + 2 \qquad f'(2) = 10(2) + 2 = 22$$

Alternative Definition of Derivative

An alternative definition of the derivative of f at point c is

$$f'(c) = \lim_{x\to c} \frac{f(x) - f(c)}{x - c},$$

provided the limit exists.



What this alternative definition allows us to do is to examine the behavior of a function as x approaches c from the left or the right. The limit exists (and thus the derivative) as long as the left and right limits exist and are equal. c is a number

$x - c = 0$
$x = c$

$$\lim_{x\to c} \frac{f(x) - f(c)}{x - c}$$

Example 3: Use the alternative definition of the derivative to find the derivative of $f(x) = \frac{1}{\sqrt{x}}$ at $x = 9$.

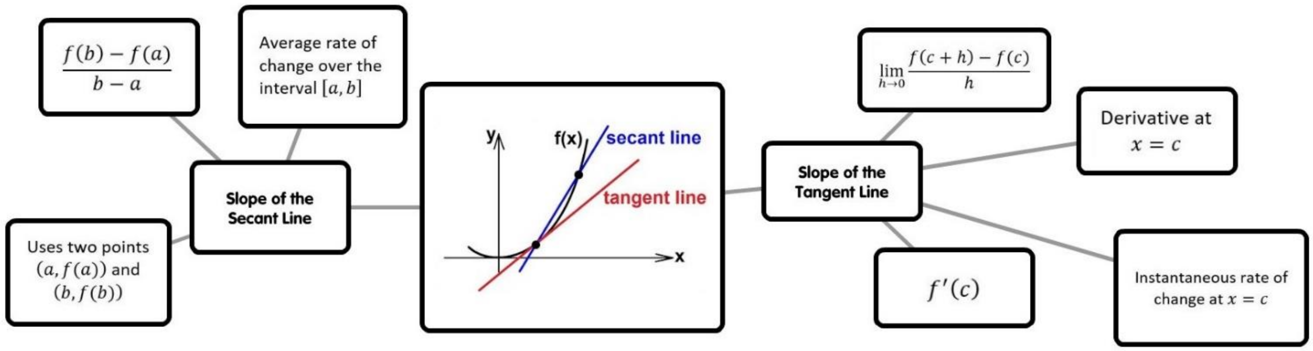
$f(x) = \frac{1}{\sqrt{x}}$

$c = 9$

$f(c) = f(9) = \frac{1}{\sqrt{9}} = \frac{1}{3}$

$$\lim_{x\to 9} \frac{\frac{3}{3}\frac{1}{\sqrt{x}} - \frac{1}{3}\frac{\sqrt{x}}{\sqrt{x}}}{x-9} \rightarrow \frac{(3-\sqrt{x})}{(3\sqrt{x})(x-9)} \cdot \frac{(3+\sqrt{x})}{(3+\sqrt{x})} = \frac{\cancel{9-x}\ -1}{3\sqrt{x}\cancel{(x-9)}(3+\sqrt{x})} \quad (3\sqrt{9})\ (3+\sqrt{9}) \rightarrow \frac{-1}{54}$$

It is important to formally lay out the meanings and distinctions between secant lines and tangent lines.



Example 4: If $f(x) = x^2 - 4x$, answer the following.

a) Find the average rate of change of $f(x)$ over the interval $[-1, 2]$.

$$\frac{f(-1)-f(2)}{-1-2}$$

$f(-1) = (-1)^2 - 4(-1) = 5$

$f(2) = 2^2 - 4(2) = -4$

$$\frac{5-(-4)}{-3} = \frac{9}{-3} = -3$$

b) Find the instantaneous rate of change of $f(x)$ at the point where $x = 2$.

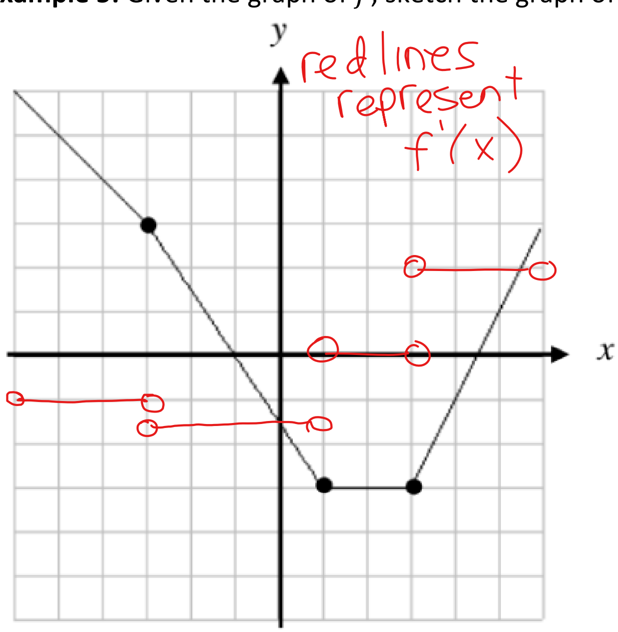
$f'(x) = 2x - 4$

$f'(2) = 4 - 4 = 0$

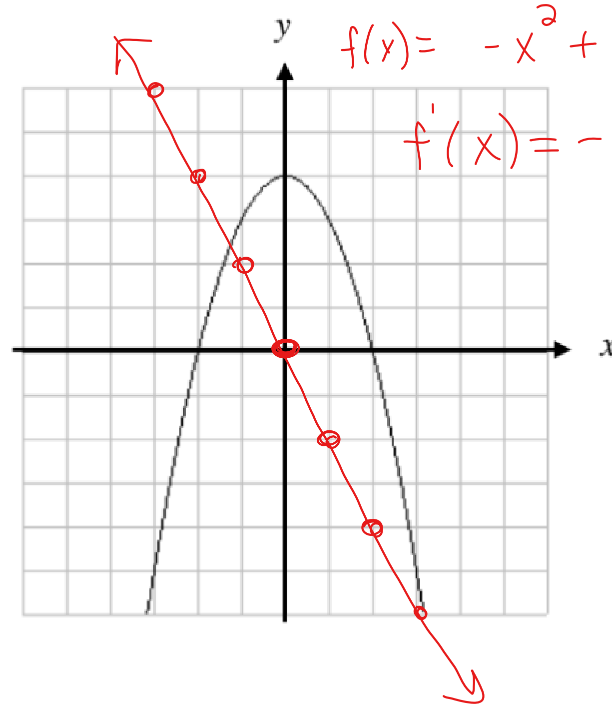
Relationships between the graphs of f and f'

Since a derivative at any point is equivalent to the slope of the function at that point, we can graph the derivative given a function and estimate what the original function looks like when we are given the graph of the derivative.

Example 5: Given the graph of f, sketch the graph of the derivative on the same set of axes.



red lines represent $f'(x)$



$f(x) = -x^2 + 4$

$f'(x) = -2x$

Example 6: Given the graph of f' and $f(0)=2$, sketch the graph of the function f on the same set of axes.

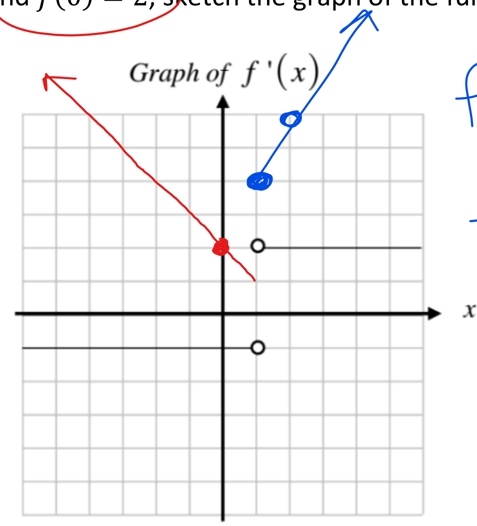
$f'(x) = -1 \quad x < 1$

$f(x) = -x + c$

$2 = -0 + c$

$2 = c$

$f(x) = -x + 2$



$f'(x) = 2 \quad x > 1$

$f(x) = 2x + c$

$2 = 2(0) + c$

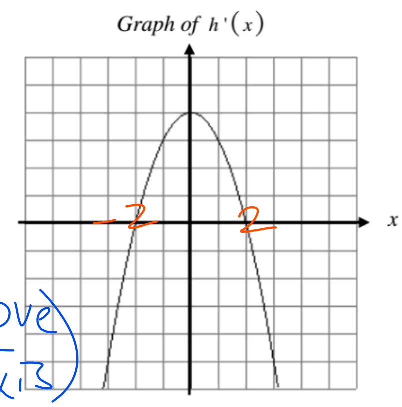
$c = 2$

$f(x) = 2x + 2$

Why is it necessary to know $f(0) = 2$?

to know "vertically" where to draw the $f(x)$

Example 7: Suppose the graph below is the graph of the derivative of h.



a) What is the value of $h'(0)$? What does this tell us about $h(x)$?

$h'(0) = 4$

↳ Inst r o c of $h(x)$ at $x=0$ is 4

b) Using the graph of $h'(x)$, how can we determine when the graph of $h(x)$ is going up? How about going down?

anytime $h'(x)$ is positive (above x-axis)

anytime $h'(x)$ is negative (below x-axis)

c) The graph of $h'(x)$ crosses the x-axis at $x = 2$ and $x = -2$. Describe the behavior of the graph of $h(x)$ at these points.

$h(x)$ is changing from decreasing to increasing at $x = -2$

and $h(x)$ is changing from increasing to decreasing at $x = 2$

AB Calculus: Differentiability **Name:**____________________________________

The focus on this section is to determine when a function fails to have a derivative. As a reminder, the word differentiable means you are able to take the derivative, or the derivative exists.

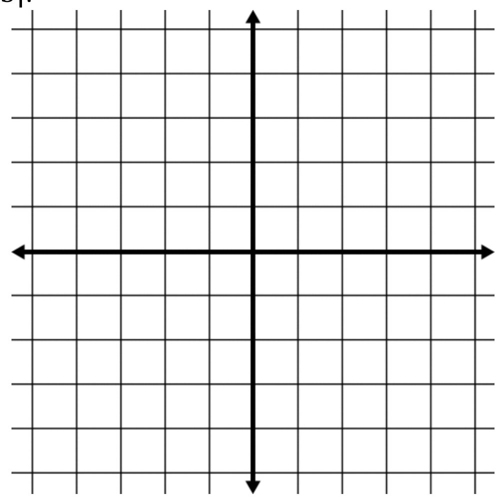
Example 1: Using the grid provided, graph the function $f(x) = |x - 3|$.

a) What is $f'(x)$ as $x \to 3^-$?

b) What is $f'(x)$ as $x \to 3^+$?

c) Is f continuous at $x = 3$?

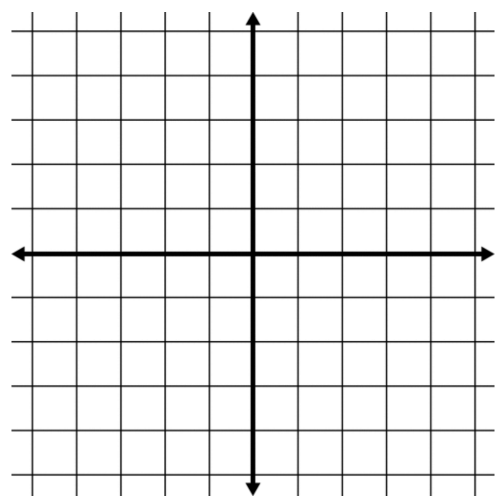
d) Is f differentiable at $x = 3$?



Example 2: Graph $f(x) = x^{\frac{2}{3}}$

a) Describe the derivative of $f(x)$ as x approaches 0 from the left and the right.

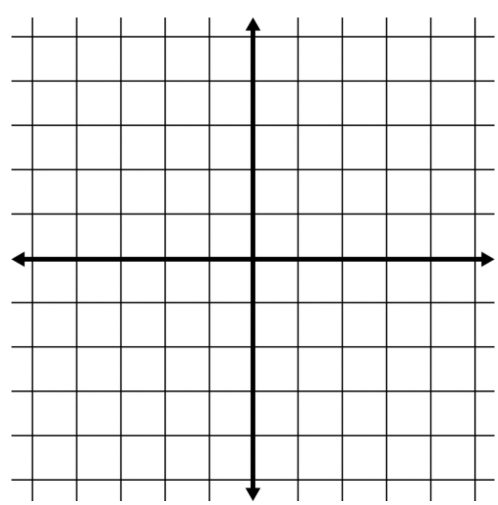
b) Suppose you found $f'(x) = \frac{2}{3\sqrt[3]{x}}$. Using this formula, what is the value of the derivative when $x = 0$?



Example 3: Graph $f(x) = \sqrt[3]{x}$

a) Describe the derivative of $f(x)$ as x approaches 0 from the left and the right.

b) Suppose you found $f'(x) = \frac{1}{3\sqrt[3]{x^2}}$. Using this formula, what is the value of the derivative when $x = 0$?



Differentiability at point $x = c$

A function $f(x)$ is differentiable at $x = c$ if and only if

$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} = L = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$$

where L is a finite value. In other words, the derivative from the left must equal the derivative from the right.

These last three examples (along with any graph that is not continuous) are not differentiable. The first graph had a **corner**, or a sharp turn, and the derivatives from the left and right were not the same. The second graph had a **cusp** where the slope approached positive infinity from one side and negative infinity from the other. The third graph had a **vertical tangent line** where the slopes approach positive or negative infinity from both sides.

You will not have to distinguish between cusp and corner. You can simply refer to them as pointy places. **Functions are not differentiable at points where the function is not continuous, at pointy places, and at points with a vertical tangent line.**

In all three of the previous examples, the functions were continuous but failed to be differentiable at certain points. Continuity does not guarantee differentiability, but differentiability does guarantee continuity.

Differentiability Implies Continuity

If f is differentiable at $x = c$, then f is continuous at $x = c$.

For the logical statement, if A, then B, the converse is written if B, then A. The converse of the statement in the box is not true! What is the converse?

The contrapositive of any statement is logically equivalent to the original statement. For the logical statement if A, then B, the contrapositive is written if not B, then not A. What is the contrapositive to the statement in the box?

Example 4: If you are given that f is differentiable at $x = 2$, then explain why each statement below is true.

a) $\lim_{x \to 2} f(x)$ exists.

b) $\lim_{h \to 0} \frac{f(2+h)-f(2)}{h}$ exists.

c) $f(2)$ exists.

d) $\lim_{x \to 2} \frac{f(x)-f(2)}{x-2}$ exists.

Example 5: If f is a function such that $\lim_{x \to -3} \frac{f(x)-f(-3)}{x+3} = 2$, which of the following must be true?

A) The limit of $f(x)$ as x approaches -3 does not exist.

B) f is not defined at $x = -3$.

C) The derivative of f at $x = -3$ is 2.

D) f is continuous at $x = 2$.

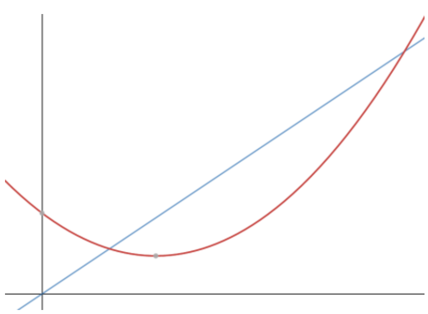
E) $f(-3) = 2$

Finding the Derivative on the Calculator

Most graphing calculators can take derivatives at certain points. In fact, it is necessary on the AP exam that you have a calculator that will take the derivative at a given point. However, the calculators use a different method of calculating the derivative that our earlier two definitions.

The Symmetric Form Definition of the Derivative
The numeric value of the derivative of a function $f(x)$ at a point $(c, f(c))$ is also given as $$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c-h)}{2h}$$

Drawing for the Symmetric Form



The TI-89 will actually find the derivative formula, but when a derivative is needed on the calculator portion of the AP Exam, it only asks you evaluate the derivative at a point, thus removing the advantage of having a TI-89 over a TI-83 or 84.

To use your calculator to find the derivative, use the nDeriv(function on the calculator). To access this function, press MATH then 8, or use ▲ ▼ to go to `8:nDeriv(` and press ENTER.

TI-83+ or TI-84 with Older Operating System	TI-84 Family with Newer Operating System
The **nDeriv(** function works as follows: **nDeriv(function, x, value)**	The **nDeriv(** function works as follows: $\frac{d}{dx}(\textbf{function})\Big\|_{x=\text{value}}$

Where **function** is the function you want to find the derivative of, **x** is the variable you are differentiating with respect to (you can use a variable other than x in the nDeriv(function if you use a different variable in your equation) and **value** is the point at which you want to find the derivative.

Note: Many times it is easier to type the function into Y_1 and then enter nDeriv(Y_1, x, value).

To enter Y_1 into the function, type VARS ► ENTER ENTER to select VARS → Y-VARS, Function, Y_1.

Example 6: Use your calculator to find the derivative of $f(x) = x^2 - 3x + 2$ at $x = -3$. Express your answer using correct notation.

Example 7: Use your calculator to find the derivative of the three examples at the beginning. What problems do you find? Why?

a) $f(x) = |x - 3|$ at $x = 3$

b) $f(x) = x^{\frac{2}{3}}$ at $x = 0$

c) $f(x) = \sqrt[3]{x}$ at $x = 0$

It takes more than just the slopes to be approaching the same value on either side of a particular x-value for a function to be differentiable there. Always be careful with piecewise functions.

Example 8: Determine if $f(x)$ is differentiable at $x = 1$ if $f(x) = \begin{cases} x^2, & x \leq 1 \\ 2x + 1, & x > 1 \end{cases}$

Example 9: Determine if $f(x)$ is differentiable at $x = 1$ if $f(x) = \begin{cases} x^2, & x \leq 1 \\ 2x - 1, & x > 1 \end{cases}$