AB Calculus: Derivatives of Inverse Functions **Name:** ______________________________

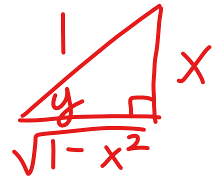
Example 1: Suppose $y = \sin^{-1} x$. Find $\frac{dy}{dx}$ using implicit differentiation.

$\sin y = x$ → take $\frac{d}{dx}$ of both sides → $\frac{d}{dx}(\sin y) = \frac{d}{dx}(x)$

$\cos y \frac{dy}{dx} = 1$ (divide by $\cos y$)

$\frac{dy}{dx} = \frac{1}{\cos y}$

$\frac{dy}{dx} = \sec y$



$\sec y$? $\left(\frac{H}{A}\right)$ $= \frac{1}{\sqrt{1-x^2}}$

Derivatives of Inverse Trig Functions where u is a function of x

$$\frac{d}{dx}[\sin^{-1} u] = \frac{1}{\sqrt{1-u^2}} \cdot u' \qquad \frac{d}{dx}[\cos^{-1} u] = \frac{-1}{\sqrt{1-u^2}} \cdot u'$$

$$\frac{d}{dx}[\tan^{-1} u] = \frac{1}{1+u^2} \cdot u' \qquad \frac{d}{dx}[\cot^{-1} u] = \frac{-1}{1+u^2} \cdot u'$$

$$\frac{d}{dx}[\sec^{-1} u] = \frac{1}{|u|\sqrt{u^2-1}} \cdot u' \qquad \frac{d}{dx}[\csc^{-1} u] = \frac{-1}{|u|\sqrt{u^2-1}} \cdot u'$$

Notes:

- Domains are restricted to make them functions so do not worry about $\sin^{-1} x$ versus $\text{Sin}^{-1} x$.
- $\sin^{-1} x$ and $\arcsin x$ are the same thing. Both refer to the inverse sine function.

Example 2: Find $\frac{d}{dt}[\sin^{-1}(t^2)]$

$= \frac{1}{\sqrt{1-(t^2)^2}} \, 2t \rightarrow \frac{2t}{\sqrt{1-t^4}}$

Example 3: Find $\frac{d}{dx}[\tan^{-1}(\sqrt{x}-1)]$

$\tan^{-1} x \rightarrow \frac{1}{1+x^2}$, $\sqrt{x} \rightarrow \frac{1}{2\sqrt{x}}$, $x-1 \rightarrow 1$

$= \frac{1}{1+(\sqrt{x-1})^2} \cdot \frac{1}{2\sqrt{x-1}} \cdot 1$

$1 + x - 1 \rightarrow x$

$= \frac{1}{2x\sqrt{x-1}}$

Example 4: Find $\frac{d}{dx}[x\sec^{-1}(3x)]$

$= x \cdot \frac{1}{|3x|\sqrt{(3x)^2-1}}(3) + 1 \sec^{-1}(3x)$

Ex 5 Given $y = x^2 + 1$, ⓐ find its inverse

$x = y^2 + 1$ (if solving for y . . . $x - 1 = y^2 \rightarrow \pm\sqrt{x-1} = y$)

ⓑ What is the slope of the original function?

$y = x^2 + 1$

$\frac{dy}{dx} = 2x$

ⓒ What is the slope of the inverse?

$\frac{d}{dx}[x = y^2 + 1]$

$1 = 2y \cdot \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{2y}$

Given $f(x) = x^3 + 2x + 1$, find $(f^{-1})'(1)$

↳ Find the derivative of the inverse of f

1 is the input of f^{-1}, therefore 1 is the output of f

$1 = x^3 + 2x + 1$

$0 = x^3 + 2x$

$0 = x(x^2 + 2)$

only real solution is $x = 0$

Now, find $f'(x) = 3x^2 + 2$

$(f^{-1})'(1) = \frac{1}{f'(0)} = \frac{1}{3(0)^2 + 2} = \frac{1}{2}$

Derivative of the Inverse Function at (p, q)

$$(f^{-1})'(p) = \frac{1}{f'(q)}$$

The derivative of $f^{-1}(x)$ at the point (p, q) is the reciprocal of the derivative of $f(x)$ at the point (q, p).

Example 6 Let $f(x) = x^5 + 2x - 1$. Find of $(f^{-1})'(-1)$ at the point $(0, -1)$ using both methods.

-1 is the input of $f^{-1}(x)$, therefore -1 is output of $f(x)$

$-1 = x^5 + 2x - 1$

$0 = x^5 + 2x$

$0 = x(x^4 + 2)$

only real solution is $x = 0$

Find $f'(x) = 5x^4 + 2$

$(f^{-1})'(-1) = \frac{1}{f'(0)} = \frac{1}{5(0)^4 + 2} = \frac{1}{2}$

Example 7 Let $f(x) = x^3 + 2x - 1$. Find $\left.\frac{df^{-1}}{dx}\right|_{x=2}$ using both methods. You can use your calculator to help you find the missing coordinate.

↳ 2 is the input of f^{-1}, SO it's the output of f.

$2 = x^3 + 2x - 1$

$0 = x^3 + 2x - 3$

1	1	0	2	-3
		1	1	3
	1	1	3	0

$0 = (x-1)(x^2 + x + 3)$

Check $b^2 - 4ac$

↳ $1^2 - 4(1)(3) < 0$

therefore there are no more real solutions

$x = 1$ is only real root

$\left.\frac{df^{-1}}{dx}\right|_{x=2} = \frac{1}{f'(1)}$

$f'(x) = 3x^2 + 2$

$f'(1) = 3(1)^2 + 2 = 5$

$\left.\frac{df^{-1}}{dx}\right|_{x=2} = \frac{1}{5}$