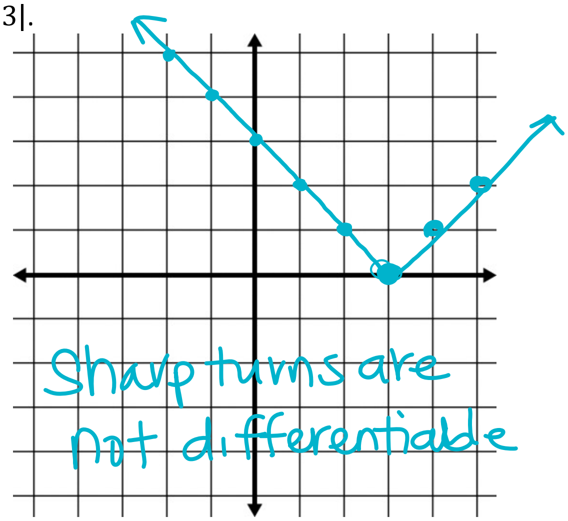
**AB Calculus: Differentiability** **Name:**_______________________

The focus on this section is to determine when a function fails to have a derivative. As a reminder, the word differentiable means you are able to take the derivative, or the derivative exists.

**Example 1:** Using the grid provided, graph the function $f(x) = |x - 3|$.



Sharp turns are not differentiable

**a)** What is $f'(x)$ as $x \to 3^-$?

what is the slope to the left of 3? $= -1$

**b)** What is $f'(x)$ as $x \to 3^+$?

" " " " " " right " " ? $= 1$

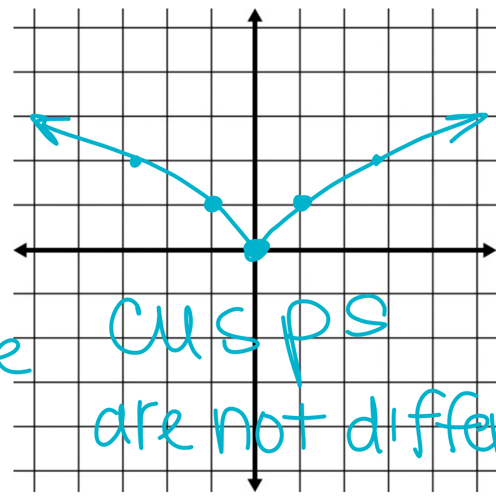
**c)** Is $f$ continuous at $x = 3$?

yes $\left( \lim_{x \to 3} f(x) = f(3) = 0 \right)$

**d)** Is $f$ differentiable at $x = 3$?

No $\lim_{x \to 3^-} f'(x) \neq \lim_{x \to 3^+} f'(x)$

**Example 2:** Graph $f(x) = x^{\frac{2}{3}}$



Cusps are not differentiable

**a)** Describe the derivative of $f(x)$ as $x$ approaches 0 from the left and the right.

$\lim_{x \to 0^-} f'(x) = -\infty$ but $\lim_{x \to 0^+} f'(x) = \infty$

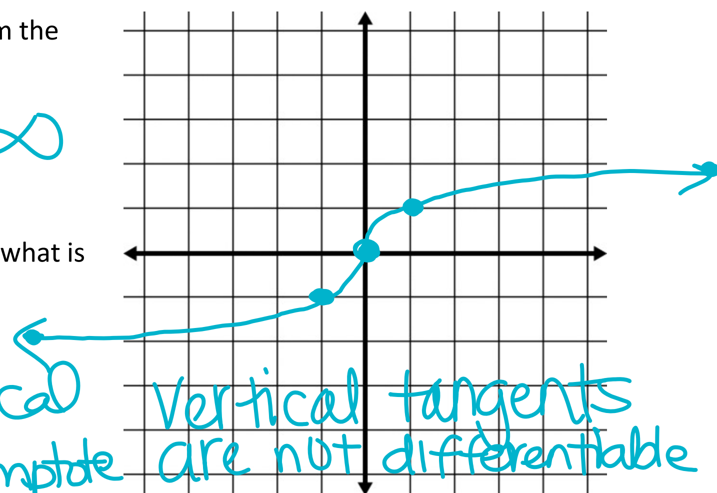
**b)** Suppose you found $f'(x) = \frac{2}{3\sqrt[3]{x}}$. Using this formula, what is the value of the derivative when $x = 0$?

$f'(0) = \frac{2}{3\sqrt[3]{0}} \to \frac{2}{0}$ vertical asymptote

$\downarrow$

undefined

**Example 3:** Graph $f(x) = \sqrt[3]{x}$



Vertical tangents are not differentiable

**a)** Describe the derivative of $f(x)$ as $x$ approaches 0 from the left and the right.

$\lim_{x \to 0^-} f'(x) = \lim_{x \to 0^+} f'(x) = \infty$

**b)** Suppose you found $f'(x) = \frac{1}{3\sqrt[3]{x^2}}$. Using this formula, what is the value of the derivative when $x = 0$?

$f'(0) = \frac{1}{3\sqrt[3]{0^2}} = \frac{1}{0} \to$ vertical asymptote

**Differentiability at point $x = c$**

A function $f(x)$ is differentiable at $x = c$ if and only if

$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} = L = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$$

where $L$ is a finite value. In other words, the derivative from the left must equal the derivative from the right.

These last three examples (along with any graph that is not continuous) are not differentiable. The first graph had a **corner**, or a sharp turn, and the derivatives from the left and right were not the same. The second graph had a **cusp** where the slope approached positive infinity from one side and negative infinity from the other. The third graph had a **vertical tangent line** where the slopes approach positive or negative infinity from both sides.

You will not have to distinguish between cusp and corner. You can simply refer to them as pointy places.
**Functions are not differentiable at points where the function is not continuous, at pointy places, and at points with a vertical tangent line.**

In all three of the previous examples, the functions were continuous but failed to be differentiable at certain points. Continuity does not guarantee differentiability, but differentiability does guarantee continuity.

**Differentiability Implies Continuity**

If $f$ is differentiable at $x = c$, then $f$ is continuous at $x = c$.

For the logical statement, if A, then B, the converse is written if B, then A. The converse of the statement in the box is not true! What is the converse?

The contrapositive of any statement is logically equivalent to the original statement. For the logical statement if A, then B, the contrapositive is written if not B, then not A. What is the contrapositive to the statement in the box?

meaning I'm allowed to take the derivative at $x = 2$

**Example 4:** If you are given that $f$ is differentiable at $x = 2$, then explain why each statement below is true.

**a)** $\lim_{x \to 2} f(x)$ exists.

Differentiability implies continuity which means $\lim_{x \to 2} f(x)$ exists

**b)** $\lim_{h \to 0} \frac{f(2+h)-f(2)}{h}$ exists.

Limit def. of a derivative ✓ (remember $x = 2$ in this example)

**c)** $f(2)$ exists.

Differentiable implies continuity so $\lim_{x \to 2} f(x) = f(2)$ already

**d)** $\lim_{x \to 2} \frac{f(x)-f(2)}{x-2}$ exists.

Alt def of derivative (at $x = 2$)

$f'(-3)=2$

**Example 5:** If $f$ is a function such that $\boxed{\lim_{x\to -3} \frac{f(x)-f(-3)}{x+3} = 2}$, which of the following must be true?

A) The limit of $f(x)$ as $x$ approaches $-3$ does not exist. False

B) $f$ is not defined at $x = -3$. False

C) The derivative of $f$ at $x = -3$ is 2.

D) $f$ is continuous at $x = 2$. Not necessarily true

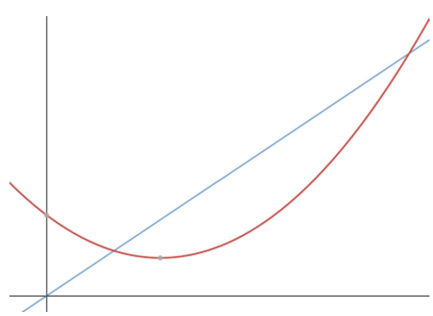
E) $f(-3) = 2$ Not necessarily true

**Finding the Derivative on the Calculator**

Most graphing calculators can take derivatives at certain points. In fact, it is necessary on the AP exam that you have a calculator that will take the derivative at a given point. However, the calculators use a different method of calculating the derivative that our earlier two definitions.

| **The Symmetric Form Definition of the Derivative** |
|---|
| The numeric value of the derivative of a function $f(x)$ at a point $(c, f(c))$ is also given as $$f'(c) = \lim_{h\to 0} \frac{f(c+h)-f(c-h)}{2h}$$ |

Drawing for the Symmetric Form



The TI-89 will actually find the derivative formula, but when a derivative is needed on the calculator portion of the AP Exam, it only asks you evaluate the derivative at a point, thus removing the advantage of having a TI-89 over a TI-83 or 84.

To use your calculator to find the derivative, use the nDeriv( function on the calculator). To access this function, press MATH then 8, or use ▲ ▼ to go to `8:nDeriv(` and press ENTER.

| TI-83+ or TI-84 with Older Operating System | TI-84 Family with Newer Operating System |
|---|---|
| The **nDeriv(** function works as follows: **nDeriv(function, x, value)** | The **nDeriv(** function works as follows: $$\frac{d}{dx}(\textbf{function})\Big\vert_{x=\text{value}}$$ |

Math → 8

Where **function** is the function you want to find the derivative of, **x** is the variable you are differentiating with respect to (you can use a variable other than x in the nDeriv( function if you use a different variable in your equation) and **value** is the point at which you want to find the derivative.

Note: Many times it is easier to type the function into $Y_1$ and then enter nDeriv($Y_1$, x, value).

To enter $Y_1$ into the function, type VARS ▶ ENTER ENTER to select VARS → Y-VARS, Function, $Y_1$.

Calculator steps

**Example 6:** Use your calculator to find the derivative of $f(x) = x^2 - 3x + 2$ at $x = -3$. Express your answer using correct notation.

math → 8 → $\frac{d}{d\boxed{x}}\left[x^2 - 3x + 2\right]\Big|_{x=\boxed{-3}}$

$f'(-3) = ?$

$f'(x) = 2x - 3$

$f'(-3) = 2(-3) - 3 = -9$

**Example 7:** Use your calculator to find the derivative of the three examples at the beginning. What problems do you find? Why?

a) $f(x) = |x - 3|$ at $x = 3$

$\frac{d}{dx}\left[|x-3|\right]\Big|_{x=3}$ → calc says 0

BUT $f'(3)$ Does not exist

Abs bar → Math → num → ABS

b) $f(x) = x^{\frac{2}{3}}$ at $x = 0$

$\frac{d}{dx}\left[x^{2/3}\right]\Big|_{x=0}$ → calc says 0 BUT $f'(0)$ Does not exist

c) $f(x) = \sqrt[3]{x}$ at $x = 0$

$\frac{d}{dx}\left[\sqrt[3]{x}\right]\Big|_{x=0}$ → calc says 100 But $f'(0)$ Does not exist

It takes more than just the slopes to be approaching the same value on either side of a particular x-value for a function to be differentiable there. Always be careful with piecewise functions.

**Example 8:** Determine if $f(x)$ is differentiable at $x = 1$ if $f(x) = \begin{cases} x^2, & x \le 1 \\ 2x + 1, & x > 1 \end{cases}$

$(1)^2 = 2(1) + 1$

$1 = 3$ $f(x)$ is not continuous at $x = 1$, therefore $f(x)$ can't be differentiable at $x = 1$

**Example 9:** Determine if $f(x)$ is differentiable at $x = 1$ if $f(x) = \begin{cases} x^2, & x \le 1 \\ 2x - 1, & x > 1 \end{cases}$

$(1)^2 = 2(1) - 1$

$1 = 1$ so $\lim_{x \to 1} f(x) = f(1) = 1$ ✓ continuous

$f'(x) = \begin{cases} 2x, & x < 1 \\ 2, & x > 1 \end{cases}$

$2(1) = 2$

✓ Differentiable at $x = 1$