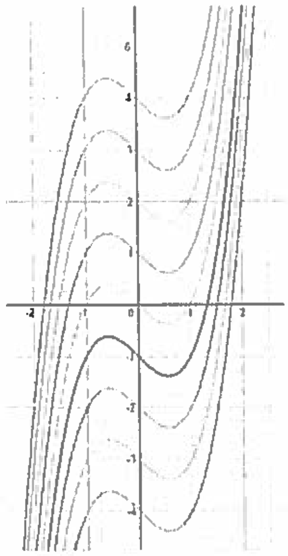
AB Calculus Differential Equations Notesheet **Name:** Luca



You have already seen that the equation $y = \int f(x)dx$ has many solutions (each differing from the others by a constant C). This means that the graph of any two antiderivatives of $f(x)$ are vertical translations of each other. The graph to the left shows several members of the family of antiderivatives $F(x) = x^3 - x + C$ for the function $f(x) = 3x^2 - 1$.

The constant C in $F(x)$ is called the **constant of integration.** $F(x)$ is called the general antiderivative of $f(x)$ and $F(x) = x^3 - x + C$ is called the **general solution** to the equation $y = \int(3x^2 - 1)dx$.

In many applications of integration, you are given enough information to determine a point on the antiderivative. This point is referred to as the **initial value or initial condition.** This information allows you to find a specific value of C and the resulting unique equation that is both the antiderivative and goes through the initial value is called the **particular solution** to the equation (in this case, $y = \int(3x^2 - 1)dx$).

Example 1 Find the general solution to the equation $f(x) = \int 6x dx$ then find the particular solution that satisfies the initial condition $f(0) = 8$.

Example 2 Find the general solution of $f(x) = \int 3x^2 dx$ then the particular solution that satisfies the initial condition $f(2) = -3$.

Differential Equations

An equation like

$$\frac{dy}{dx} = \frac{x^3}{y}$$

containing a derivative is called a differential equation. The **order** of a differential equation is the order of the highest derivative involved in the equation. The problem of finding a function y of x when given its derivative and its value at a particular point is called an **initial value problem**. The value of f for one value of x is the initial condition of the problem. When all the functions y that satisfy the differential equation have been found, then the differential equation has been solved. When the particular solution that fulfills the initial condition has been found, then the initial value problem has been solved.

Examples of differential equations

$$\frac{dy}{dx} = \frac{x^2}{2y} \qquad \frac{d^2y}{dx^2} = \cos x \qquad \frac{dy}{dx} = 3x + 2$$

The previous examples are all called **separable differential equations** because it is possible to separate all the x and y variables. When given a separable differential equation in Leibniz form $\left(\frac{dy}{dx}\right)$, it is **mandatory** to show the separation of variables by rewriting the function in differentiable form. If $\frac{dy}{dx} = f(x)$, then

$dy = f(x)dx$ is the differentiable form.

The process of finding the antiderivatives of each side of the above equation is called indefinite integration. We can denote this operation with an integral symbol. By taking the integral of both sides of the differential form to find the general solution, we get

$$\int dy = \int f(x)\, dx$$

$$y = F(x) + C$$

Solving Differential Equations Using Separation of Variables

1. Move everything "y" to the left (including dy) and everything "x" to the right (including dx). Keep constants on the right if possible.
2. Integrate both sides and add $+C$ to the "x" side. (You do not need to add constants to both sides because you would just combine the constants together which would just produce a constant).
3. If you are given a point, plug-in the point and solve for C, then substitute the value you found in for C.
4. Solve the equation for y.

Example 3 Solve $\frac{dy}{dx} = \sin x$ using separation of variables if $y(0) = 2$.

Example 4 The acceleration of a body moving along a coordinate line can be modeled by the function $a(t) = \cos t$. Find the velocity and position functions if $v(0) = -1$ and $s(0) = 1$.

Example 5 Find the general solution for $\frac{dy}{dx} = e^x + 20(1 + x^2)^{-1}$ then the particular solution that satisfies the initial condition $y(0) = -2$.

Example 6 Find the general solution for $\frac{dy}{dx} = 3y$ then the particular solution that satisfies the initial condition $y(0) = 1$.

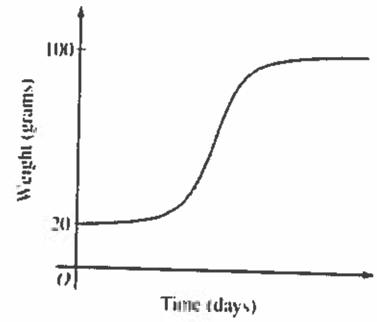
The rate at which a baby bird gains weight is proportional to the difference between its adult weight and its current weight. At time $t = 0$, when the bird is first weighed, its weight is 20 grams. If $B(t)$ is the weight of the bird, in grams, at time t days after it is first weighed, then

$$\frac{dB}{dt} = \frac{1}{5}(100 - B).$$

Let $y = B(t)$ be the solution to the differential equation above with initial condition $B(0) = 20$.

(a) Is the bird gaining weight faster when it weighs 40 grams or when it weighs 70 grams? Explain your reasoning.

(b) Find $\frac{d^2B}{dt^2}$ in terms of B. Use $\frac{d^2B}{dt^2}$ to explain why the graph of B cannot resemble the following graph.



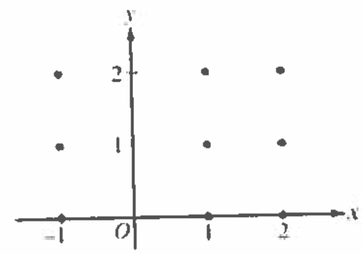
(c) Use separation of variables to find $y = B(t)$, the particular solution to the differential equation with initial condition $B(0) = 20$.

Consider the differential equation $\frac{dy}{dx} = \left(1 - \frac{2}{x^2}\right)(y - 1)$, where $x \neq 0$.

Let $y = f(x)$ be the particular solution to the differential equation with initial condition $f(1) = 2$.

(a) Find the slope of the line tangent to the graph of f at the point $(1, 2)$.

(b) On the axes provided, sketch a slope field for the given differential equation at the nine points indicated.



(c) Find the particular solution $y = f(x)$ to the differential equation $\frac{dy}{dx} = \left(1 - \frac{2}{x^2}\right)(y - 1)$ with initial condition $f(1) = 2$.

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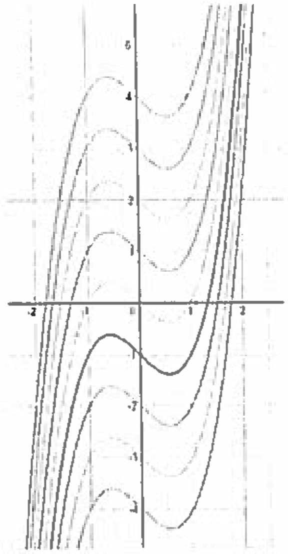

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Example 1 Find the general solution to the equation $f(x) = \int 6x dx$ then find the particular solution that satisfies the initial condition $f(0) = 8$.

$\frac{6x^2}{2} + c$ or $3x^2 + c$

$8 = 3(0)^2 + c$

$y = 3x^2 + 8$

Example 2 Find the general solution of $f(x) = \int 3x^2 dx$ then the particular solution that satisfies the initial condition $f(2) = -3$.

$\frac{3x^3}{3}$ or $x^3 + c$

$-3 = (2)^3 + c$

$-3 = 8 + c$

$c = -11$

$y = x^3 - 11$

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Examples of differential equations

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Example 3 Solve $\frac{dy}{dx} = \sin x$ using separation of variables if $y(0) = 2$.

$\int dy = \int \sin x\, dx$ $\quad y = -\cos x + c$ $\quad 2 = -\cos 0 + c$ $\quad 2 = -1 + c$ $\quad c = 3$ $\quad \boxed{y = -\cos x + 3}$

Example 4 The acceleration of a body moving along a coordinate line can be modeled by the function $a(t) = \cos t$. Find the velocity and position functions if $v(0) = -1$ and $s(0) = 1$.

$v(t) = \sin t + c$ $\quad \boxed{v(t) = \sin t - 1}$

$-1 = \sin 0 + c$

$c = -1$

$s(t) = -\cos t - t + c$

$1 = -\cos 0 - 0 + c$

$1 = -1 + c$

$c = 2$ $\quad \boxed{s(t) = -\cos t - t + 2}$

Example 5 Find the general solution for $\frac{dy}{dx} = e^x + 20(1 + x^2)^{-1}$ then the particular solution that satisfies the initial condition $y(0) = -2$.

$\int dy = \int \left(e^x + \frac{20}{1+x^2}\right) dx$ $\quad y = e^x + 20\tan^{-1} x + c$

$-2 = e^0 + 20\tan^{-1} 0 + c$

$-2 = 1 + 0 + c$ $\quad c = -3$

$\boxed{y = e^x + 20\tan^{-1} x -}$

Example 6 Find the general solution for $\frac{dy}{dx} = 3y$ then the particular solution that satisfies the initial condition $y(0) = 1$.

$\frac{dy}{dx} = 3y$ $\quad \int \frac{dy}{y} = \int 3dx$

$\ln|y| = 3x + c$

$\ln|1| = 3(0) + c$

$0 = c$

$\ln|y| = 3x$

$y = e^{3x}$

The rate at which a baby bird gains weight is proportional to the difference between its adult weight and its current weight. At time $t = 0$, when the bird is first weighed, its weight is 20 grams. If $B(t)$ is the weight of the bird, in grams, at time t days after it is first weighed, then

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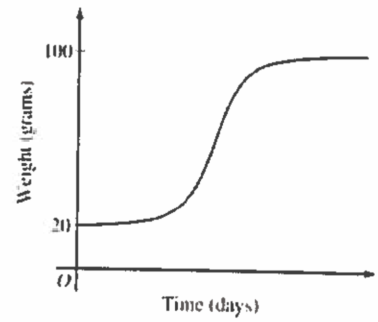


(c) Use separation of variables to find $y = B(t)$, the particular solution to the differential equation with initial condition $B(0) = 20$.

(a) $\left.\frac{dB}{dt}\right|_{B=40} = \frac{1}{5}(100-40) = \frac{1}{5}(60) = 12$

$\left.\frac{dB}{dt}\right|_{B=70} = \frac{1}{5}(100-70) = \frac{1}{5}(30) = 6$

faster when weighs 40 grams b/c $\frac{dB}{dt}$ when $B=40$ is larger than $\frac{dB}{dt}$ when $B=70$.

(b) $\frac{d^2B}{dt^2} = \frac{1}{5}(100-B)^0\left(-\frac{dB}{dt}\right) = -\frac{1}{5}\left(\frac{1}{5}\right)(100-B)$ $\to B=100$ $-\ |\ +$

The 2nd derivative means the concavity of B(t) should change Concave down to cu when, B=100 and the graph's concavity changes around B=50.

(c) $\int \frac{dB}{100-B} = \int \frac{1}{5}\,dt$

$-\ln|100-B| = \frac{1}{5}t + C$

$-\ln|100-20| = \frac{1}{5}(0) + C$

$-\ln 80 = C$

$-\ln|100-B| = \frac{1}{5}t - \ln 80$

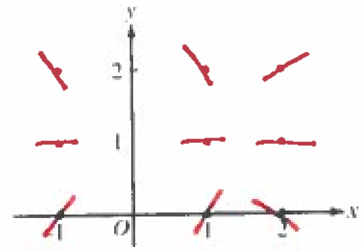
$\ln|100-B| = -\frac{1}{5}t + \ln 80$

$100 - B = 80e^{-\frac{1}{5}t}$

$-B = 80e^{-\frac{1}{5}t} - 100$

$\boxed{B(t) = -80e^{-\frac{1}{5}t} + 100}$

Consider the differential equation $\frac{dy}{dx} = \left(1 - \frac{2}{x^2}\right)(y-1)$, where $x \neq 0$.



Let $y = f(x)$ be the particular solution to the differential equation with initial condition $f(1) = 2$.

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(c) Find the particular solution $y = f(x)$ to the differential equation $\frac{dy}{dx} = \left(1 - \frac{2}{x^2}\right)(y-1)$ with initial condition $f(1) = 2$.

(a) $\left.\frac{dy}{dx}\right|_{(1,2)} = \left(1 - \frac{2}{1^2}\right)(2-1) = (1-2)(1) = -1$

(b) slope field

(c) $\int \frac{dy}{y-1} = \int \left(1 - \frac{2}{x^2}\right) dx$ $\quad 2x^{-2}$

$\ln|y-1| = x + \frac{2}{x} + C$

$|y-1| = ce^{x + \frac{2}{x}}$

$1 = ce^{1+2}$

$1 = ce^{3}$

$c = e^{-3}$

$|y-1| = e^{-3} \cdot e^{x + \frac{2}{x}}$

$\boxed{y = e^{x + \frac{2}{x} - 3} + 1}$