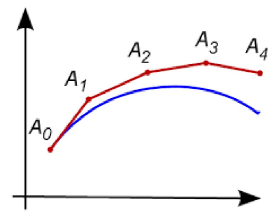
AB Calculus Euler's Method Notesheet Name: ______________________________

For differential equations that cannot be solved, a slope field provides a graphical solution to the differential equation. The problem with this approach is that a slope field is only really good for getting general trends in solutions and for long-term behavior of solutions. There are times when we will need something more. For example, it is sometimes useful to know how a specific solution behaves, including some values that the solution will take.

In these cases, we must resort to numerical methods that will allow us to **approximate** the solutions to differential equations. There are many different methods that can be used to approximate solutions to a differential equation. We have already seen one example, linearization, otherwise known as tangent line approximations, which can give us a decent approximation of a solution near a point of tangency. Today, we are going to look at a method devised by 18th century mathematician Leonhard Euler that expands upon this idea.



Euler's method basically involves walking out along a tightrope from an initial point along its tangent line. Instead of walking along the same line the whole time as in a tangent line approximation, we change tangent lines with each step (of length dx). This involves recalculating the point and slope after each step. This will produce a much more accurate approximation than simply using the original tangent line. The process itself is pretty easy and repetitive, and it is easier demonstrated with an example rather than a complicated formula. First, a bit of background information.

- We will need to designate the number of equal steps we would like to take. Call this number n.
- To find dx, the length of each step, we calculate how far we want to go and divide by how many steps we want to take. Assume $x = a$ is our initial x-value and $x = b$ is the x-value we are trying to get to. Then $dx = \frac{b-a}{n}$.
- Recall slope $m = \frac{dy}{dx}$. If you solve for dy, we get $dy = m(dx)$. (Just cross-multiply

We can now proceed. The following chart will make things easier to organize. Commit it to memory.

old pt	dx	$m = \left.\frac{dy}{dx}\right\|_{(x,y)}$	dy = m dx	new pt

$dx = \frac{8-0}{2} = 4$

Example 1 Given the differential equation $\frac{dy}{dx} = x - 2$ and $y(0) = 5$. m → old pt

a) Find an approximation for $y(0.8)$ by using Euler's method with two equal steps.

old pt	dx	$m = \frac{dy}{dx}$	dy = m dx	new pt
(0,5)	.4	0-2=-2	(-2)(4) = -8	(4, 42)
(4,42)	.4	4-2=-16	(-16)(4) = -64	(.8, 356)

$4\cancel{2}.\cancel{0}10 - 64 = 356$

b) Solve the differential equation $\frac{dy}{dx} = x - 2$ with the initial condition $y(0) = 5$, and use your solution to find $y(0.8)$.

$\frac{dy}{dx} = x - 2$

cross multiply

$\int dy = \int (x-2)\,dx$

$y = \frac{x^2}{2} - 2x + C$

$5 = \frac{0^2}{2} - 2(0) + C$

$5 = C$

$\boxed{y = \frac{x^2}{2} - 2x + 5}$

$y(.8) = \frac{.8^2}{2} - 2(.8) + 5 = 3.72$

$y(.8) \approx 3.56$

$dx = \frac{1.7-2}{3} = \frac{-.3}{3} = -.1$

Example 2 If $\frac{dy}{dx} = 2x - y$ and $y = 3$ when $x = 2$, use Euler's method with 3 equal steps to approximate y when $x = 1.7$.

old pt	dx	$m = \frac{dy}{dx}$	$dy = m\ dx$	new pt
$(2,3)$	-0.1	$2(2)-3=1$	$1\cdot -1 = -.1$	$(1.9,2.9)$
$(1.9,2.9)$	-0.1	$2(1.9)-2.9=.9$	$.9\cdot -.1 = -.09$	$(1.8,2.81)$
$(1.8,2.81)$	-0.1	$2(1.8)-2.81=.79$	$.79 \cdot -.1 = -.079$	$(1.7,2.731)$

$\boxed{y(1.7) \approx 2.731}$

Example 3 Assume that f and f' have the values given in the table. Use Euler's method with two equal steps to approximate the value of $f(2.6)$.

$\frac{2.6-3}{2} = \frac{-.4}{2} = -.2$

x	3	2.8	2.6
$\frac{dy}{dx}$ $f'(x)$	0.4	0.7	~~0.9~~
$f(x)$	2	1.92	1.78

old pt	dx	$m = \frac{dy}{dx}$	$dy = m\ dx$	new pt
$(3,2)$	$-.2$	$.4$	$(.4)(-.2) = -0.08$	$(2.8, 1.92)$
$(2.8,1.92)$	$-.2$	$.7$	$(.7)(-.2) = -.14$	$(2.6, 1.78)$

$\boxed{f(2.6) \approx 1.78}$

Example 4 Consider the differential equation $\frac{dy}{dx} = y^2(2x+2)$. Let $y = f(x)$ be the particular solution to the differential equation with initial condition $f(0) = -1$.

a) Find $\lim_{x \to 0} \frac{f(x)+1}{\sin x}$. Show the work that leads to your answer.

$\lim_{x\to 0} \frac{f(0)+1}{\sin(0)} = \lim_{x \to 0} \frac{-1+1}{0}$

L'Hospitals $\lim_{x\to 0} \frac{f(x)+1}{\sin x} = \lim_{x\to 0} \frac{f'(x)}{\cos x} = \frac{f'(0)}{\cos 0} = \frac{(-1)^2(2\cdot 0+2)}{1} = \frac{2}{1} = 2$

b) Use Euler's Method, starting at $x = 0$ with two steps of equal size, to approximate $f\left(\frac{1}{2}\right)$. $f(\frac{1}{2}) \approx \frac{-11}{32}$

$\frac{\frac{1}{2}-0}{2} = \frac{1}{4}$

old pt.	dx	$m = \frac{dy}{dx}$	$dy = m\ dx$	new pt.
$(0,-1)$	$\frac{1}{4}$	$(-1)^2(2\cdot 0+2) = 2$	$(2)(\frac{1}{4}) = \frac{1}{2}$	$(\frac{1}{4}, -\frac{1}{2})$
$(\frac{1}{4}, -\frac{1}{2})$	$\frac{1}{4}$	$(-\frac{1}{2})^2(2\cdot\frac{1}{4}+2) = \frac{1}{4}\cdot\frac{5}{2} = \frac{5}{8}$	$(\frac{5}{8})(\frac{1}{4}) = \frac{5}{32}$	$(\frac{1}{2}, \frac{-1}{2}+\frac{5}{32})$

$\boxed{-\frac{1}{2}+\frac{5}{32} \quad \frac{-16}{32}+\frac{5}{32} = -\frac{11}{32}}$

c) Find $y = f(x)$, the particular solution to the differential equation with initial condition $f(0) = -1$.

$\frac{dy}{dx} = y^2(2x+2)$

$\frac{dy}{y^2} = (2x+2)\,dx$

$\int y^{-2}\,dy = \int (2x+2)\,dx$

$\frac{y^{-1}}{-1} = x^2 + 2x + C$

$\frac{(-1)^{-1}}{-1} = 0^2 + 2(0) + C$

$1 = C$

$\frac{y^{-1}}{-1} = x^2 + 2x + 1$

$y^{-1} = -1(x^2+2x+1)$

$\frac{1}{y} = -1(x^2+2x+1) \rightarrow \boxed{y = \frac{1}{-1(x^2+2x+1)}}$