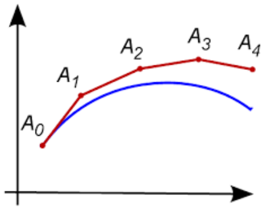
AB Calculus Euler's Method Notesheet **Name:** ______________________________

For differential equations that cannot be solved, a slope field provides a graphical solution to the differential equation. The problem with this approach is that a slope field is only really good for getting general trends in solutions and for long-term behavior of solutions. There are times when we will need something more. For example, it is sometimes useful to know how a specific solution behaves, including some values that the solution will take.

In these cases, we must resort to numerical methods that will allow us to **approximate** the solutions to differential equations. There are many different methods that can be used to approximate solutions to a differential equation. We have already seen one example, linearization, otherwise known as tangent line approximations, which can give us a decent approximation of a solution near a point of tangency. Today, we are going to look at a method devised by 18th century mathematician Leonhard Euler that expands upon this idea.



Euler's method basically involves walking out along a tightrope from an initial point along its tangent line. Instead of walking along the same line the whole time as in a tangent line approximation, we change tangent lines with each step (of length dx). This involves recalculating the point and slope after each step. This will produce a much more accurate approximation than simply using the original tangent line. The process itself is pretty easy and repetitive, and it is easier demonstrated with an example rather than a complicated formula. First, a bit of background information.

- We will need to designate the number of equal steps we would like to take. Call this number n.
- To find dx, the length of each step, we calculate how far we want to go and divide by how many steps we want to take. Assume $x = a$ is our initial x-value and $x = b$ is the x-value we are trying to get to. Then $dx = \frac{b-a}{n}$.
- Recall slope $m = \frac{dy}{dx}$. If you solve for dy, we get $dy = m(dx)$.

We can now proceed. The following chart will make things easier to organize. Commit it to memory.

Old pt (x_1, y_1)	dx	$m = \frac{dy}{dx}$	$dy = m \cdot dx$	New pt (x_2, y_2)
$(0,5)$	$.4$	$m = 0-2 = -2$	$(-2)(.4) = -.8$	$(.4, 4.2)$
$(.4, 4.2)$	$.4$	$m = .4-2 = -1.6$	$(-1.6)(.4) = -.64$	$(.8, 3.56)$

$\frac{.8-0}{2} = .4$
dx

Example 1 Given the differential equation $\frac{dy}{dx} = x - 2$ and $y(0) = 5$.

$\boxed{y(.8) \approx 3.56}$

a) Find an approximation for $y(0.8)$ by using Euler's method with two equal steps.

b) Solve the differential equation $\frac{dy}{dx} = x - 2$ with the initial condition $y(0) = 5$, and use your solution to find $y(0.8)$.

$\int dy = \int (x-2)\,dx$

$y = \frac{1}{2}x^2 - 2x + C$

$5 = \frac{1}{2}(0)^2 - 2(0) + C$

$C = 5$

$\boxed{y = \frac{1}{2}x^2 - 2x + 5}$

$\boxed{y(.8) = 3.72}$

plug in to x

$\frac{1.7-2}{3} = \frac{-.3}{3} = -.1$ $y(1.7) \approx 2.731$

Example 2 If $\frac{dy}{dx} = 2x - y$ and $y = 3$ when $x = 2$, use Euler's method with 3 equal steps to approximate y when $x = 1.7$.

old pt.	dx	$m = \frac{dy}{dx}$	$dy = m\,dx$	new pt
$(2,3)$	$-.1$	$2(2)-3=1$	$1\cdot -.1 = -.1$	$(1.9, 2.9)$
$(1.9, 2.9)$	$-.1$	$2(1.9)-2.9 = .9$	$.9\cdot -.1 = -.09$	$(1.8, 2.81)$
$(1.8, 2.81)$	$-.1$	$2(1.8)-2.81 = .79$	$.79\cdot -.1 = -.079$	$(1.7, 2.731)$

Example 3 Assume that f and f' have the values given in the table. Use Euler's method with two equal steps to approximate the value of $f(2.6)$. $-.2$

$f(2.6) \approx 1.78$

x	3	2.8	2.6
$f'(x)$	0.4	0.7	~~0.9~~ → never use
$f(x)$	2	1.92	1.78

old pt	dx	$\frac{dy}{dx} = m$	$dy = m\cdot dx$	new pt.
$(3,2)$	$-.2$	$.4$	$(.4)(-.2) = -.08$	$(2.8, 1.92)$
$(2.8, 1.92)$	$-.2$	$.7$	$(.7)(-.2) = -.14$	$(2.6, 1.78)$

→ remember $= f'(x)$

Example 4 Consider the differential equation $\frac{dy}{dx} = y^2(2x+2)$. Let $y = f(x)$ be the particular solution to the differential equation with initial condition $f(0) = -1$.

a) Find $\lim_{x\to 0} \frac{f(x)+1}{\sin x}$. Show the work that leads to your answer.

$\lim_{x\to 0} f(x)+1 = 0$

$\lim_{x\to 0} \sin x = 0$

$\lim_{x\to 0} \frac{f(0)+1}{\sin 0} \to \frac{-1+1}{0} \to \frac{0}{0}$

L'Hospital's

$\to \lim_{x\to 0} \frac{f(x)+1}{\sin x} = \lim_{x\to 0} \frac{f'(x)}{\cos x} \to \frac{f'(0)}{\cos 0} \to \frac{(-1)^2(2\cdot 0+2)}{1} = 2$

b) Use Euler's Method, starting at $x = 0$ with two steps of equal size, to approximate $f\left(\frac{1}{2}\right)$.

old pt	dx	$m = \frac{dy}{dx}$	$dy = m\,dx$	new pt
$(0,-1)$	$.25$	$(-1)^2(2\cdot 0+2) = 2$	$2\cdot .25 = .50$	$(.25, -.50)$
$(.25, -.50)$	$.25$	$(-.5)^2(2\cdot .25+2) = .625$	$.625\cdot .25 = .15625$	$(.50, -11/32)$

c) Find $y = f(x)$, the particular solution to the differential equation with initial condition $f(0) = -1$.

$\frac{dy}{dx} = y^2(2x+2)$

$\int \frac{dy}{y^2} = \int (2x+2)\,dx$

$-\frac{1}{y} = x^2 + 2x + C$

$\frac{-1}{-1} = 0^2 + 0 + C$

$C = 1$

$-\frac{1}{y} = x^2 + 2x + 1$

$y = -\frac{1}{x^2+2x+1}$