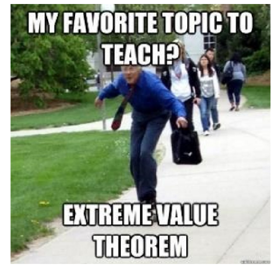
AB Calculus: Extreme Values of a Function **Name:** ____________________

Extrema (plural for extremum) are the maximum and minimum values of a function. In the past, you have used your calculator to calculate the maximum and minimum value. In this section, you will learn to use calculus reasons to find extrema, how to distinguish between absolute extrema and relative extrema, and how to locate them. continuous on closed interval $[a, b]$

Definition of Absolute Extrema … the BIGGEST or smallest y-value in the interval

Let f be defined on an interval I containing c

1. $f(c)$ is the minimum of f on the interval I if $f(c) \leq f(x)$ for all x in I.
2. $f(c)$ is the maximum of f on the interval I if $f(c) \geq f(x)$ for all x in I.



The minimum and maximum of a function on an interval are the **extreme values**, or **extrema**, of the function on the interval. The minimum and maximum of a function on an interval are also called the **absolute minimum** and **absolute maximum**, or the **global minimum** and **global maximum**, on the interval. Extrema can occur at interior points or endpoints of an interval.

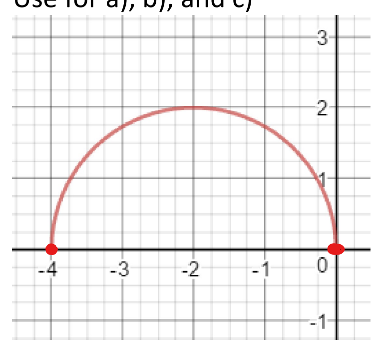
A function may have both a maximum and a minimum value over an interval, only a maximum, only a minimum, or neither a maximum or minimum value of an interval. Try the following example to see some reasons why.

$(x+2)^2 + (y-0)^2 = 4$

Example 1 Find the absolute extrema of the functions over each given interval. Use the top graph for parts a,b, and c and use the bottom graph for part d.

$y = \sqrt{4-(x+2)^2}$

Use for a), b), and c)



a) $[-4, 0]$ abs max at $(-2, 2)$

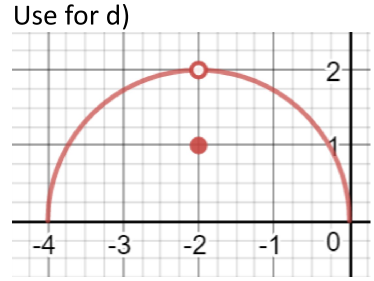
abs min at $(-4, 0)$, $(0, 0)$

b) $[-2, 0)$ abs max at $(-2, 2)$

No abs min

c) $(-4, -2)$ no abs min nor max

Use for d)



d) $[-4, 0]$ [not continuous]

abs min at $(-4, 0)$, $(0, 0)$

no max

What two things needed to be the case for there to be both a minimum and a maximum for the graphs in Ex. 1?

The Extreme Value Theorem

If f is continuous over a closed interval $[a, b]$, then f has both a minimum and a maximum over the interval.

Relative Extrema and Critical points

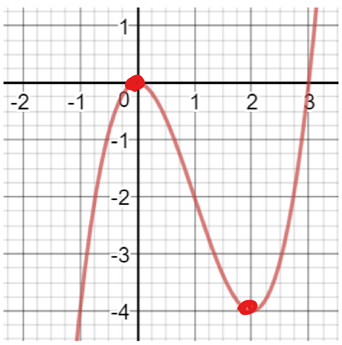
Definition of Relative Extrema

local

1. If there is an open interval containing a c on which $f(c)$ is a maximum, then $f(c)$ is called a **relative maximum** or **local maximum**. You could also say that f has a relative maximum at $(c, f(c))$.
2. If there is an open interval containing a c on which $f(c)$ is a minimum, then $f(c)$ is called a **relative minimum** or **local minimum**. You could also say that f has a relative minimum at $(c, f(c))$.

Basically, relative extrema exist when the value of the function is larger (or smaller) than all other function values relatively close to that value.

Example 2: Locate the relative extrema in each graph below and determine the value of the derivative.



Relative Extrema:

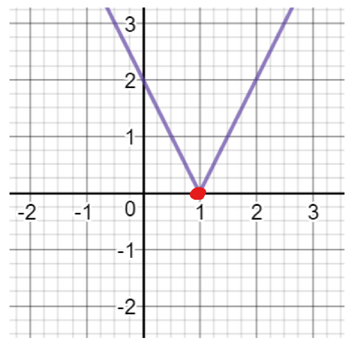
at $(0,0)$ local max

at $(2,-4)$ local min

What is the value of the derivative at each Relative Extrema?

$f'(x) = 0$

at $x = 0, 2$



Relative Extrema:

at $(1,0)$

local min

What is the value of the derivative at each Relative Extrema?

$f'(x)$ at $x=1$

Does not exist

Definition of Critical Point

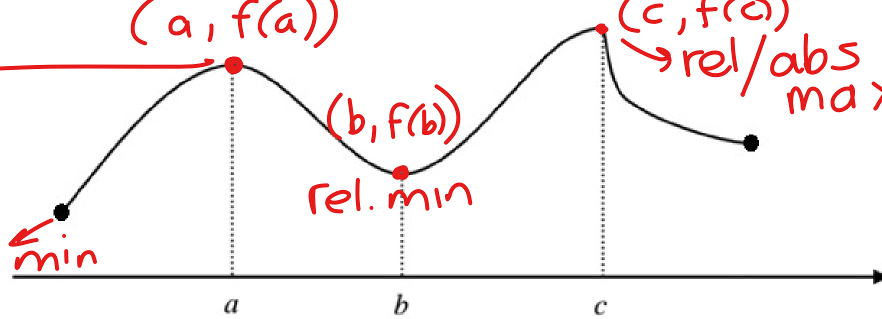
Let f be defined at c. If $f'(c) = 0$ or if $f'(c)$ is undefined, then c is a **critical point** of f.

Relative Extrema only occur at critical points

If f has a relative max or relative min at $x = c$, then c is a critical point of f.

Example 3 the graph of $f(x)$ is shown below. Label $f(a)$ through $f(c)$ as absolute or relative extrema.

relative max



$(a, f(a))$

$(c, f(c))$ rel/abs max

$(b, f(b))$ rel. min

abs min

When given a graph it is fairly simple to identify the extrema. The question to be asked then is how do we find the extrema when we do not have a graph given to us.

Important Note: Just because the derivative is equal to zero (or undefined) does not mean there is a relative maximum or minimum at the point. The sign of the derivative needs to change sign for a maximum or minimum to exist at that point. This does not happen at every critical point, but it can only happen at critical points.

critical pts or end pts

Guidelines for Finding Absolute/Relative Extrema on a Closed Interval

critical pts only

1. Find the critical numbers of f on (a, b). Do this by setting the derivative equal to 0 and undefined. These critical points and the endpoints make up the list of candidates for the extrema.
2. Evaluate each candidate by plugging these numbers into the original function.
3. The least of the values from the previous step is the absolute minimum, and the greatest of these values is the absolute maximum.

The critical points are <u>x</u> values, while maximums/minimums of the function are <u>y</u> values. In other words, if the point $(2, 70)$ is a relative minimum, the minimum of the function is <u>70</u> and it occurs at <u>2</u>.

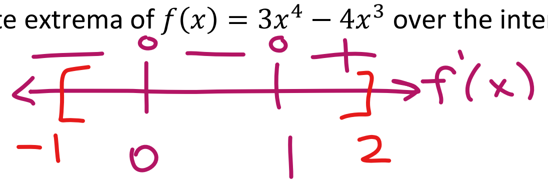
Example 4 Find the absolute extrema of $f(x) = 3x^4 - 4x^3$ over the interval $[-1, 2]$.

$f'(x) = 12x^3 - 12x^2$

$0 = 12x^2(x-1)$

$x = 0$, $x = 1$

these are my critical pts



abs min is $\underline{-1}$ at $x = 1$

$f(-1) = 3 + 4 = 7$

$f(2) = 48 - 32 = 16$

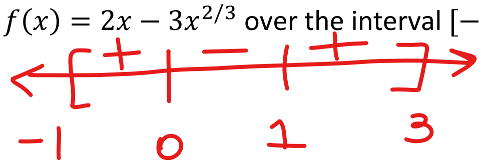
SO abs max is 16 at $x = 2$

Example 5 Find the extrema of $f(x) = 2x - 3x^{2/3}$ over the interval $[-1, 3]$.

$f'(x) = 2 - 2x^{-\frac{1}{3}}$

$f'(x) = 2 - \dfrac{2}{\sqrt[3]{x}}$

$0 = 2 - \dfrac{2}{\sqrt[3]{x}}$



if $x = 1$, then $f'(1) = 0$

if $x = 0$, then $f'(0)$ is undefined

$f(-1) = -2 - 3 = -5$

$f(0) = 0$

$f(1) = 2 - 3 = -1$

$f(3) = 6 - 3\sqrt[3]{9}$

abs max is 0 at $x = 0$

abs min is -5 at $x = -1$

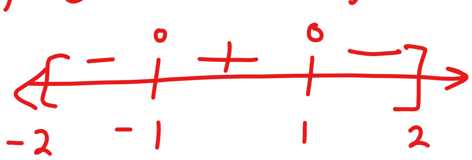
Example 6 Find the absolute extrema of $f(x) = \frac{2x}{x^2+1}$ over the interval $[-2, 2]$.

$f'(x) = \dfrac{(x^2+1)(2) - (2x)(2x)}{(x^2+1)^2}$

$= \dfrac{2x^2 + 2 - 4x^2}{(x^2+1)^2}$

$= \dfrac{-2x^2 + 2}{(x^2+1)^2} = \dfrac{-2(x^2-1)}{(x^2+1)^2} = \dfrac{-2(x+1)(x-1)}{(x^2+1)^2}$

$f'(x) = 0$ when $x = 1, -1 \rightarrow$ critical pts



$f(-2) = \frac{-4}{5}$

$f(-1) = \frac{-2}{2}$

$f(1) = \frac{2}{2}$

$f(2) = \frac{4}{5}$

abs max is 1 at $x = 1$

abs min is -1 at $x = -1$