BC Calculus Improper Integrals Day 1 Homework **Name:** Key

1. Determine whether the integral converges or diverges, and evaluate the integral if it converges.

a) $\int_0^{\infty} e^{-3x}\,dx$ $\lim_{b\to\infty}\int_0^b e^{-3x}\,dx \to \lim_{b\to\infty} -\frac{1}{3}e^{-3x}\Big|_0^b \to -\frac{1}{3}\left(\lim_{b\to\infty}(e^{-3b} - e^0)\right)$

$\to -\frac{1}{3}\left(\lim_{b\to\infty}\left(\frac{1}{e^{3b}} - 1\right)\right) \to -\frac{1}{3}(0-1) \to \frac{1}{3}$ convergent

b) $\int_1^{\infty} \frac{dx}{\sqrt{x}}$ → p-series w/ $p \le 1$ $(p = \frac{1}{2})$ should diverge

$\lim_{b\to\infty}\int_1^b x^{-\frac{1}{2}}\,dx \to \lim_{b\to\infty} 2x^{\frac{1}{2}}\Big|_1^b \to \lim_{b\to\infty} 2\sqrt{b} - 2 \to \infty - 2 \to \infty$ so it diverges

2. Evaluate the integral or state that it diverges.

a) $\int_1^{\infty} \frac{dx}{x^{1.001}}$ since $a=1$ & $p>1$ converges to $\frac{1}{1.001-1} = 1000$

b) $\int_{-\infty}^{-2} \frac{1}{x+1}\,dx$ $\lim_{a\to-\infty}\int_a^{-2} \frac{1}{x+1}\,dx \to \lim_{a\to-\infty}\left(\ln|x+1|\right)\Big|_a^{-2} \to$

$\to \lim_{a\to-\infty}\left(\ln|-2+1| - \ln|a+1|\right) \to \lim_{a\to-\infty}\left(0 - \ln|a+1|\right)$

$-1+1 \mid$

$\to -\ln|-\infty+1| \to -\ln\infty \to \infty$

divergent

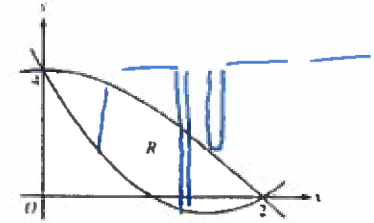
c) $\int_1^{\infty} \frac{6}{x^4}\,dx$ $a=1$ $p>1$ so p-series

$\frac{1}{4-1} = \frac{1}{3}$ but don't forget 6 → $6\left(\frac{1}{3}\right) = 2$

d) $\int_3^{\infty} \frac{1}{x^2}\,dx$ $a \ne 3$ so not the pattern $\frac{1}{p-1}$

$\lim_{b\to\infty}\int_3^b x^{-2}\,dx \to \lim_{b\to\infty} \frac{-1}{x}\Big|_3^b \to \frac{-1}{\infty} + \frac{1}{3} \to \frac{1}{3}$

3. Let $f(x) = 2x^2 - 6x + 4$ and $g(x) = 4\cos\left(\frac{1}{4}\pi x\right)$. Let R be the region bounded by the graphs of f and g, as shown in the figure to the right.



a) Find the area of R.

$$\text{Area} = \int_0^2 (g(x) - f(x))\,dx \quad (\approx 3.760)$$

b) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = 4$.

$$V = \pi \int_0^2 \left((4 - f(x))^2 - (4 - g(x))^2\right) dx$$

c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a square. Write, but do not evaluate, an integral expression that gives the volume of this solid.

$$V = \int_0^2 (g(x) - f(x))^2\,dx$$

4. Evaluate each of the following integrals.

a) $\int \frac{x^2}{e^x}dx \rightarrow \int x^2 e^{-x} dx$

$u = x^2$ + $dv = e^{-x}dx$

$2x$ − $-e^{-x}$

2 + e^{-x}

0 $-e^{-x}$

$$\boxed{-x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C}$$

b) $\int \frac{\ln x}{x}dx$

$u = \ln x$ $\quad du = \frac{1}{x}dx$

$\int u\,du \rightarrow \frac{u^2}{2} + C$

$$\boxed{\rightarrow \frac{(\ln x)^2}{2} + C}$$

c) $\int \frac{\ln x}{x^2}dx$ $\quad u = \ln x \quad dv = x^{-2}dx$

$du = \frac{1}{x}dx \quad v = -x^{-1}\ (-\frac{1}{x})$

$\left(-\frac{1}{x}\right)(\ln x) - \int -\frac{1}{x}\cdot\frac{1}{x}\,dx$

$-\frac{\ln x}{x} + \int \frac{1}{x^2}dx$

$$\rightarrow -\frac{\ln x}{x} - \frac{1}{x} + C$$

d) $\int \frac{4x}{\sqrt{1-x^2}}dx$

$u = 1 - x^2$ $\quad -\frac{1}{2}du = x\,dx$

$\frac{du}{dx} = -2x$

$\rightarrow 4\left(-\frac{1}{2}\right)\int \frac{1}{\sqrt{u}}du \rightarrow -2\int u^{-\frac{1}{2}}du$

$$\rightarrow -2\cdot 2u^{\frac{1}{2}} + C = -4\sqrt{1-x^2} + C$$

e) $\int \cos(2x)e^{-x}\,dx$

$u = e^{-x}$ $\quad dv = \cos(2x)dx$

$du = -e^{-x}dx$ $\quad v = \frac{1}{2}\sin(2x)$

new u & dv

$u = e^{-x}$ $\quad dv = \sin(2x)dx$

$du = -e^{-x}dx$ $\quad v = -\frac{1}{2}\cos(2x)$

$\frac{1}{2}e^{-x}\sin(2x) + \int \frac{1}{2}\sin(2x)\cdot(+e^{-x})dx$

$\rightarrow -\frac{1}{2}e^{-x}\cos(2x) - \int +\frac{1}{2}\cos(2x)(+e^{-x})dx$

Final steps: $\int \cos(2x)e^{-x}dx = \frac{1}{2}e^{-x}\sin(2x) + \frac{1}{2}\left(-\frac{1}{2}e^{-x}\cos(2x) - \frac{1}{2}\int \cos(2x)e^{-x}dx\right)$

final coeff → $-\frac{1}{4}$ so add $\frac{1}{4}$ to 1 = $\frac{5}{4}$ → divide by $\frac{5}{4}$ → $\frac{4}{5}$ → $\frac{2}{5}e^{-x}\sin(2x) - \frac{1}{5}e^{-x}\cos(2x) + C$