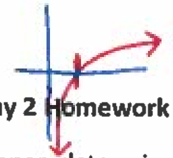
Remember $\ln x$  so $\ln 0^+$ tends to $-\infty$

**AP Calculus Improper Integrals Day 2 Homework** Name: Key

1. State why the integral is improper, determine whether the integral converges or diverges, and evaluate the integral if it converges.

a) $\int_{-8}^{0} \frac{dx}{x^{\frac{1}{3}}}$ There is an asymptote @ $x=0$

$\lim_{b\to 0^-} \int_{-8}^{b} x^{-\frac{1}{3}}dx \to \lim_{b\to 0^-} \frac{3}{2}x^{\frac{2}{3}}\Big|_{-8}^{b} \to \frac{3}{2}(0)^{\frac{2}{3}} - \frac{3}{2}(-8)^{\frac{2}{3}}$

$\to -\frac{3}{2}(4) = -6 \to$ converges

b) $\int_{2}^{4} \frac{dx}{x-2}$ V.A. @ $x=2$

$\lim_{a\to 2^+} \int_{a}^{4} \frac{dx}{x-2} \to \lim_{a\to 2^+} \ln|x-2|\Big|_{a}^{4} \to \ln(4-2) - \ln(a-2)$

$\to \ln 2 - \ln(2.000000001-2)$ $-\infty$

$\to \ln 2 + \infty \to$ diverge

2. Evaluate the integral or state that it diverges.

a) $\int_{0}^{4} \frac{\ln x}{x}dx$ $\lim_{a\to 0^+} \int_{a}^{4} \frac{\ln x}{x}dx$ $u=\ln x$ $du=\frac{1}{x}dx$ $u=\ln 4$

$\lim_{a\to 0^+} \int_{\ln a}^{\ln 4} u\,du \to \lim_{a\to 0^+} \frac{u^2}{2}\Big|_{\ln a}^{\ln 4} \to \frac{(\ln 4)^2}{2} - \cancel{\frac{(\ln a)^2}{2}}$ $-\infty$ $=+\infty$ $\to$ so diverges

b) $\int_{0}^{1} \frac{dx}{\sqrt{1-x^2}}$ $\lim_{b\to 1^-} \int_{0}^{b} \frac{1}{\sqrt{1-x^2}}dx \to \lim_{b\to 1^-} \sin^{-1}x\Big|_{0}^{b} \to \lim_{b\to 1^-} \sin^{-1}b - \sin^{-1}0$

$\to \frac{\pi}{2} - 0 = \frac{\pi}{2}$

c) $\int_{1}^{\infty} \frac{1}{x^5}dx$ $a=1$ $p>1$ so converges b/c p-series

$\frac{1}{5-1} = \frac{1}{4}$

d) $\int_{-\infty}^{-2} \left[\frac{1}{x-1} - \frac{1}{x+1}\right]dx$

V.A @ 1 V.A @ $-1$

$\lim_{a\to -\infty} \int_{a}^{-2} \frac{1}{x-1} - \frac{1}{x+1} \to \lim_{a\to -\infty} \left(\ln|x-1|\Big|_{a}^{-2} - \ln|x+1|\Big|_{a}^{-2}\right)$

$\lim_{a\to -\infty} (\ln|-2-1| - \ln|a-1| - \ln|-2+1| + \ln|a+1|)$

$\to \ln 3 - \ln\infty - \ln 1 + \ln\infty$

$\to$ Indeterminate

so instead do $\ln\left|\frac{x-1}{x+1}\right|$

$\to \ln\left|\frac{3}{1}\right| - \ln\left|\frac{a-1}{a+1}\right|$

$\to \ln 3 - \ln 1 = \ln 3$

3. Evaluate the following integrals

a) $\int 3x \ln x \, dx$

$u = \ln x \qquad dv = 3x\,dx$

$du = \frac{1}{x}dx \qquad v = \frac{3}{2}x^2$

$\int 3x\ln x\,dx = \frac{3}{2}x^2 \ln x - \int \frac{3}{2}x^2 \cdot \frac{1}{x}\,dx = \boxed{\frac{3}{2}x^2 \ln x - \frac{3}{4}x^2 + C}$

b) $\int 5x^2 \cos(4x) dx$

| | | |
|---|---|---|
| $u = 5x^2$ | + | $dv = \cos 4x$ |
| $10x$ | − | $\frac{1}{4}\sin 4x$ |
| $10$ | + | $-\frac{1}{16}\cos 4x$ |
| $0$ | | $-\frac{1}{64}\sin 4x$ |

$\boxed{= \frac{5}{4}x^2 \sin(4x) + \frac{5}{8}x\cos(4x) - \frac{5}{32}\sin(4x) + C}$

c) $\int x \sec^2(4x^2) dx$

$u = 4x^2$

$du = 8x\,dx$

$\hookrightarrow = \frac{1}{8}\int \sec^2 u\,du$

$= \frac{1}{8}\tan u + C \boxed{= \frac{1}{8}\tan(4x^2) + C}$

d) $\int \left(x^3 - \frac{1}{x} + \frac{2}{x^2} + \sqrt[4]{x}\right) dx$

$\frac{2}{x^2} \to 2x^{-2} \qquad \sqrt[4]{x} \to x^{\frac{1}{4}}$

$\boxed{= \frac{x^4}{4} - \ln|x| - \frac{2}{x} + \frac{4}{5}x^{\frac{5}{4}} + C}$

e) $\int \frac{dx}{x \ln x}$

$u = \ln x$

$du = \frac{1}{x}dx$

$\hookrightarrow = \int \frac{1}{u}du = \ln|u| + C$

$\boxed{= \ln|\ln x| + C}$

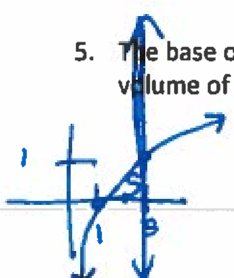
f) $\int \frac{dx}{x^2 \sqrt[3]{x}} \rightarrow \int x^{-\frac{7}{3}}dx$

$\sqrt[3]{x} \to x^{1/3}$

$\boxed{= -\frac{3}{4}x^{-\frac{4}{3}} + C}$

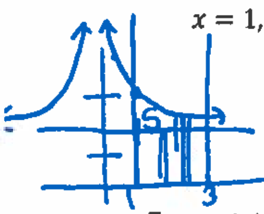
4. Find the length of the curve $y = x^3$ from $x = 0$ to $x = 2$. Ignore for now

5. The base of a solid $S$ is the region enclosed by the graph of $y = \ln x$, the line $x = e$, and the x-axis. Find the volume of the solid if the cross sections of $S$ taken perpendicular to the x-axis are squares.




$V = \int_1^e (\ln x - 0)^2\,dx \boxed{\approx .718}$

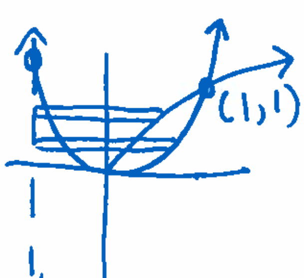
6. Find the volume of the solid formed by revolving the region in the first quadrant bounded by the graphs of $y = \frac{1}{x^2}$, $x = 1$, and $x = 3$ about the line $y = -2$.




$V = \pi \int_1^3 \left(\left(\frac{1}{x^2} + 2\right)^2 - (0+2)^2\right) dx \boxed{\approx 9.386}$

7. Find the volume of the solid generated by revolving the region bounded by the graphs of $y = \sqrt{x}$ and $y = x^2$ about the line $x = -3$.

$y^2 = x \qquad \sqrt{y} = x$




$V = \pi \int_0^1 \left((\sqrt{y} + 3)^2 - (y^2 + 3)^2\right) dx \boxed{\approx 7.226}$