AP Calculus Improper Integrals Day 3 Homework   Name: Key

1. Determine whether the integral converges or diverges, and evaluate the integral if it converges.

a) $\int_1^\infty \frac{dx}{x^4}$

converges p-series $a=1$, $p=4$

converges to $\frac{1}{4-1}=\frac{1}{3}$

b) $\int_0^2 \frac{1}{x^3}dx$

$\lim_{a\to 0^+}\int_a^2 x^{-3}dx \to \lim_{a\to 0^+}\left(\frac{-1}{2}x^{-2}\Big|_a^2\right) \to \lim_{a\to 0^+} \frac{-1}{2(2)^2}+\frac{1}{2a^2}$

$\to -\frac{1}{8}+\infty$ diverges

c) $\int_1^\infty \frac{\ln x}{x}dx$

$u=\ln x \quad du=\frac{1}{x}dx \qquad \lim_{b\to\infty}\int_1^b \frac{\ln x}{x}dx$

$\int u\,du \to \frac{u^2}{2} \to \frac{(\ln x)^2}{2} \qquad \to \lim_{b\to\infty}\left(\frac{\ln x}{2}\right)^2\Big|_1^b$

$\to \lim_{b\to\infty} \frac{(\ln b)^2}{2}-\left(\frac{\ln 1}{2}\right)^2 \to \infty - 0$

diverges

d) $\int_1^\infty \frac{1}{\sqrt{3x-1}}dx$

$u=3x-1 \to du=3dx$

$\frac{du}{3}=dx$

$\lim_{b\to\infty}\int_1^b \frac{1}{\sqrt{3x-1}}dx \to \lim_{b\to\infty}\left[\frac{2}{3}\sqrt{3x-1}\right]_1^b$

$\frac{1}{3}\int u^{-\frac{1}{2}}du \to \frac{1}{3}(2)u^{\frac{1}{2}} \to \frac{2}{3}\sqrt{u}$

$\to \lim_{b\to\infty} \frac{2}{3}\sqrt{3b-1}-\frac{2}{3}\sqrt{3(1)-1} \to \infty - \frac{2}{3}\sqrt{2}$

diverges

e) $\int_0^4 \frac{1}{(4-x)^{3/2}}dx$

$-\int u^{-3/2}du$

$\hookrightarrow \frac{+2}{\sqrt{u}} \to \frac{2}{\sqrt{4-x}}$

$u=4-x$

$\frac{du}{dx}=-1$

$-du=dx$

$\lim_{a\to 4^-}\int_0^a \frac{1}{(4-x)^{3/2}}dx \to \lim_{a\to 4^-}\left[\frac{2}{\sqrt{4-x}}\right]_0^a$

$\to \lim_{a\to 4^-}\left[\frac{2}{\sqrt{4-a}}-\frac{2}{\sqrt{4-0}}\right] \to \infty - 1$

diverges

f) $\int_{-\infty}^\infty \frac{x}{(1+x^2)^2}dx$

$u=1+x^2$

$du=2x\,dx$

$\frac{1}{2}du=x\,dx$

$\lim_{a\to-\infty}\int_a^0 \frac{x}{(1+x^2)^2}dx + \lim_{b\to\infty}\int_0^b \frac{x}{(1+x^2)^2}dx$

$\frac{1}{2}\int u^{-2}du \to \frac{-1}{2u}$

$\lim_{a\to-\infty}\left[\frac{-1}{2(1+x^2)}\right]_a^0 + \lim_{b\to\infty}\left[\frac{-1}{2(1+x^2)}\right]_0^b$

converges

$\lim_{a\to-\infty}\left[\frac{-1}{2(1+0^2)}+\frac{1}{2(1+a^2)}\right]+\lim_{b\to\infty}\left[\frac{-1}{2(1+b^2)}+\frac{1}{2(1+0^2)}\right] \to -\frac{1}{2}+0-0+\frac{1}{2} \to \boxed{0}$

2. Find the volume of the solid formed by revolving the unbounded region lying between the graphs of $y = \frac{1}{x}$, $x = 1$, and the x-axis around the x-axis.

$V = \pi \int_1^{\infty} \left(\frac{1}{x}\right)^2 dx \rightarrow \pi \int_1^{\infty} \frac{1}{x^2} dx$

↳ p-series w/ a=1 and p=2

converges to $\frac{1}{2-1} = \frac{1}{1} = 1$

so $\pi \cdot 1 = \pi$

3. Use the direct comparison test to determine whether $\int_1^{\infty} \frac{1}{x^5+1} dx$ converges or diverges.

$\frac{1}{x^5+1} \leq \frac{1}{x^5}$ for $x \geq 1$

$\int_1^{\infty} \frac{1}{x^5} dx \rightarrow$ p-series $a=1$, $p=5$ converges to $\frac{1}{5-1} = \frac{1}{4}$

since $\frac{1}{x^5+1} \leq \frac{1}{5}$ and $\int_1^{\infty} \frac{1}{x^5} dx$ converges, $\int_1^{\infty} \frac{1}{x^5+1} dx$ converges by DCT.

4. Use the direct comparison test to determine whether $\int_2^{\infty} \frac{1}{\sqrt{x-1}} dx$ converges or diverges.

$\frac{1}{\sqrt{x-1}} \geq \frac{1}{\sqrt{x}}$ for $x \geq 2$

$\int_2^{\infty} \frac{1}{x^{\frac{1}{2}}} dx \rightarrow$ since $p \leq 1$ this will diverge

↳ p-series

since $\frac{1}{\sqrt{x-1}} \geq \frac{1}{\sqrt{x}}$ and $\int_2^{\infty} \frac{1}{\sqrt{x}} dx$ diverges, $\int_2^{\infty} \frac{1}{\sqrt{x-1}} dx$ diverges by DCT.

5. Evaluate the following integrals.

a) $\int x^3\sqrt{4+x^4}\,dx$

$u = 4+x^4$

$\frac{du}{dx} = 4x^3 \rightarrow \frac{du}{4} = x^3 dx$

$\frac{1}{4}\int u^{\frac{1}{2}} du$

↳ $\frac{1}{4} \cdot \frac{2}{3} u^{\frac{3}{2}} \rightarrow \boxed{\frac{1}{6}(4+x^4)^{\frac{3}{2}} + C}$

b) $\int x^2 \ln x\, dx$

$u = \ln x$ $\quad dv = x^2 dx$

$du = \frac{1}{x} dx$ $\quad v = \frac{x^3}{3}$

$\frac{x^3}{3}\ln x - \int \frac{x^3}{3} \cdot \frac{1}{x} dx$ $\quad \frac{1}{3}\int x^2 dx \rightarrow \frac{1}{9}x^3$

$\boxed{\frac{x^3}{3}\ln x - \frac{1}{9}x^3 + C}$

c) $\int x^2 e^{5x} dx$

$u = x^2$ $\quad dv = e^{5x} dx$

$2x$ $\quad \frac{1}{5}e^{5x}$

2 $\quad \frac{1}{25}e^{5x}$

0 $\quad \frac{1}{125}e^{5x}$

$\boxed{\frac{1}{5}x^2 e^{5x} - \frac{2}{25}xe^{5x} + \frac{2}{125}e^{5x} + C}$

d) $\int x 3^{x^2} dx$

$u = x^2$ $\quad \frac{du}{2} = x\,dx$

$\frac{1}{2}\int 3^u du \rightarrow \boxed{\frac{1}{2} \cdot \frac{1}{\ln 3} \cdot 3^{x^2} + C}$

e) $\int \frac{5x^2 + 6\sqrt{x}}{4x^3} dx$

$\int \frac{5x^2}{4x^3} + \int \frac{6x^{1/2}}{4x^3}$

$\frac{5}{4}\int \frac{1}{x} dx + \frac{3}{2}\int x^{-\frac{5}{2}} dx$

$\boxed{\frac{5}{4}\ln|x| - x^{-\frac{3}{2}} + C}$

f) $\int \frac{3}{x\sqrt{x^2-1}} dx$

$\rightarrow 3\int \frac{1}{x\sqrt{x^2-1}} dx$

$\rightarrow \boxed{3\sec^{-1} x + C}$