AB Calculus: Intro to Limits **Name:** Key

The limit is fundamental to the study of calculus. It is important to acquire a good working knowledge of the limit before moving forward, because you will find out through the duration of this course that really, it is all about limits.

KNOW YOUR LIMITS

Example 1: Use your calculator to generate a graph of $f(x) = \frac{x^2-4}{x-2};\ x \neq 2$.

a) Why is $x \neq 2$ included in the function definition?

Because it's a hole.

b) Complete the table of values below to determine what happens as x gets "close" to 2.

	x approaches 2 from the left →					← x approaches 2 from the right				
x	1.5	1.75	1.9	1.99	1.999	2	2.001	2.01	2.1	2.25	2.5
$f(x)$	3.5	3.75	3.9	3.99	3.999	DNE	4.001	4.01	4.1	4.25	4.5

Definition of a Limit

If the y-value of a function $f(x)$ becomes close to a single value L as x gets closer and closer to a point c from both the left and the right side, then the limit of $f(x)$ as x approaches c is L, which can be written using mathematical notation as:

$$\lim_{x \to c} f(x) = L$$

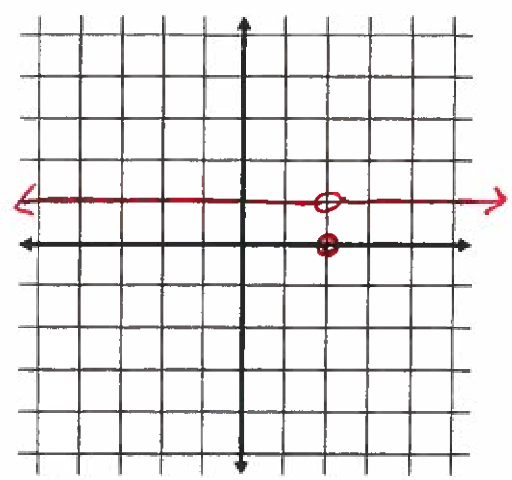
c) Apply this definition to the function from above to find $\lim_{x \to 2} f(x)$. $= 4$

The previous definition of a limit is an informal definition. The formal mathematical definition is called the epsilon-delta definition. It is not required for AP Calculus, however, if you would like to learn about it, click on, or type in, the following link or scan the QR code to the right.
https://goo.gl/gyzBaK

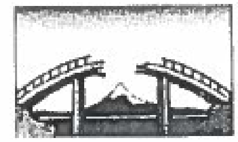
Example 2: Use the graph to find $\lim_{x \to 2} g(x)$, where g is:

$$g(x) = \begin{cases} 1, & x \neq 2 \\ 0, & x = 2 \end{cases}$$

$= 1$

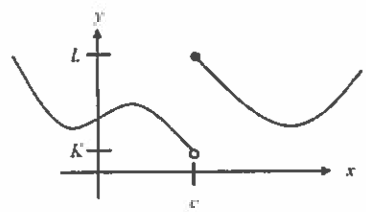


The last example can be confusing at first, but think of a damaged bridge crossing a canyon. Even though the bridge is broken and it seems like the missing section has fallen into the water below, you can probably tell where the missing section should be. The height of that point would be the value of the limit. Limits are important because they give us the ability to discuss what is going on at a point mathematically whether the bridge, or graph, exists at that point or not. With limits, you are only concerned with the y-value the graph approaches, not what the y-value actually is at that point.



One-Sided Limits

Suppose we have the graph of $f(x)$ below. Notice that the function below does not approach the same y-value as x approaches c from the left and right sides. When tracing the graph starting to the left of c, the graph approaches the y-value K. When tracing the graph starting to the right of c, the graph approaches the y-value L.



Sometimes we are only interested in what the function approaches as x approaches from the right or left of c. We can say this using the following notation:

$\lim_{x\to c^+} f(x) = L$ ... "the limit of $f(x)$ as x approaches c from the right is L."
$\lim_{x\to c^-} f(x) = K$ ... "the limit of $f(x)$ as x approaches c from the left is K."

The limit of a function as x approaches any number c exists if and only if the limit as x approaches c from the right is equal to the limit as x approaches c from the left. Using limit notation, we have:

$$\boxed{\lim_{x\to c} f(x) \text{ exists} \Longleftrightarrow \lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x)}$$

Example 3: Let $f(x) = \begin{cases} 5-2x, & x>2 \\ x-3, & x\le 2 \end{cases}$ $5-4=1$ $2-3=-1$

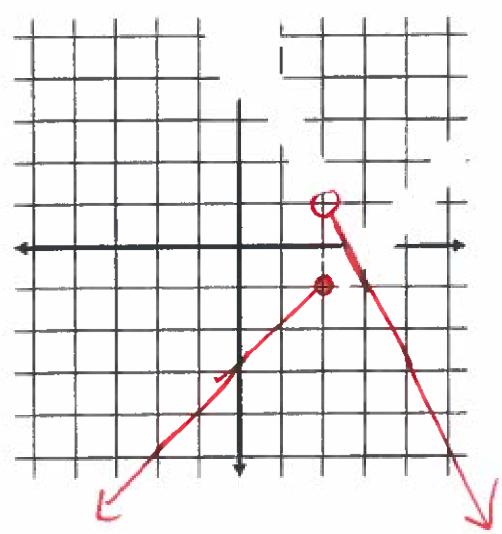
a) Graph $f(x)$.

b) Find $\lim_{x\to 2^-} f(x)$. $=1$

c) Find $\lim_{x\to 2^+} f(x)$. $=+1$

d) Find $\lim_{x\to 2} f(x)$? DNE

Limit from Left
$\neq$ LFR



When Limits Do Not Exist

There are times when a limit fails to exist, meaning that we are unable to find a value for the limit at the given point. When this happens, we say that the limit **does not exist**, abbreviating this statement as **DNE**. There are three types of common function behavior that lead to a limit that does not exist.

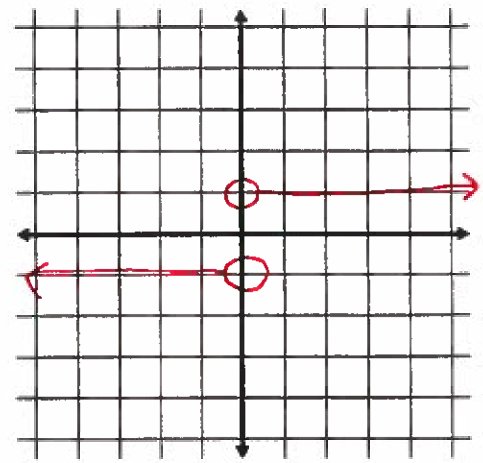
Common Types of Behavior Associated with a Limit that Does Not Exist
1. $f(x)$ approaches a different number from the right side of c than it approaches from the left side. 2. $f(x)$ increases or decreases without bound as x approaches c. 3. $f(x)$ oscillates between two fixed values as x approaches c.

Note: We will be more specific about Cause Number 2 when we explore infinite limits.

Example 4: Investigate the existence of the following limit using a graph and table.

$\lim_{x \to 0} \frac{|x|}{x}$

$$f(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$$



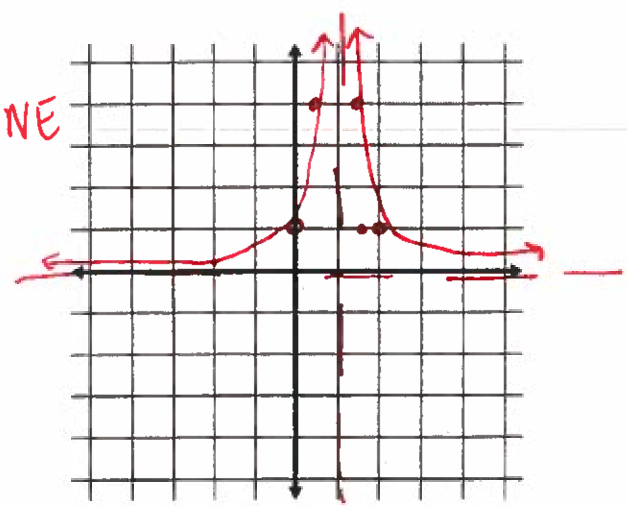
x	-0.5	-0.25	-0.1	-.01	-.001	0	.001	.01	.1	.25	.5
$f(x)$	-1	-1	-1	-1	-1	DNE	1	1	1	1	1

Limit DNE

Example 5: Investigate the existence of the following limit using a graph and table.

Technically DNE

$\lim_{x \to 1} \frac{1}{(x-1)^2}$ ∞



x	0.5	0.75	0.9	0.99	0.999	1	1.001	1.01	1.1	1.25	1.5
$f(x)$	4	16	100	10,000	1000000	DNE	1,000,000	10,000	100	16	4

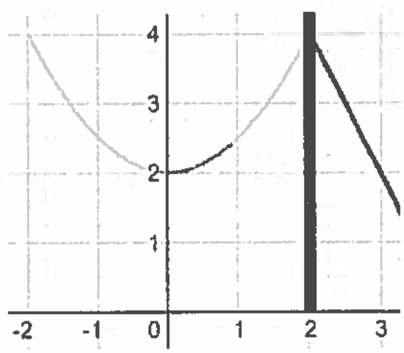
Example 6: Use a table to investigate $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$

x	$\frac{2}{\pi}$	$\frac{2}{3\pi}$	$\frac{2}{5\pi}$	$\frac{2}{7\pi}$	$\frac{2}{9\pi}$	$\frac{2}{11\pi}$	$\frac{2}{13\pi}$	As $x \to 0$
$f(x)$	1	-1	1	-1	1	-1	1	DNE - oscillates

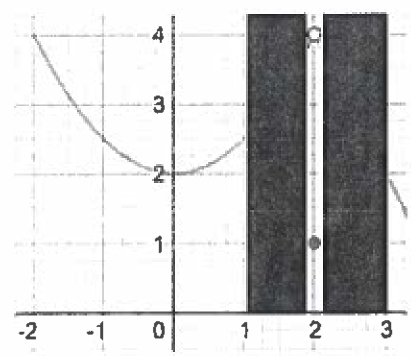
Finding Limits from Graphs

Hopefully you are getting comfortable with this new idea of a limit value and how it is different from a function value. When trying to find limits and function values from graphs, refer to the following visual aids:

When looking for a limit value at $x = c$, imagine that you have got a thick vertical line covering up $x = c$ with only the graph showing on either side of $x = c$. You are now looking to see what y-value the graph is approaching on either side of $x = c$. If the graphs appear to be approaching the same y-value, the limit exists and is that y-value. Otherwise, the limit does not exist.



When looking for a function value at $x = c$, imagine that you have got shudders on either side of $x = c$ with only a vertical sliver at $x = c$ visible between them. You are now looking for the dot or the piece of graph that exists in that narrow sliver. If it exists, the y-value of the dot is the function value $f(c)$. Otherwise, the function value is undefined.



Example 7: Use the graph to evaluate the following limits

A) $\lim_{x \to 0} f(x)$ 6

B) $\lim_{x \to 6^-} f(x)$ 4

C) $\lim_{x \to 2^-} f(x)$ 0

D) $\lim_{x \to -6} f(x)$ DNE
NEI

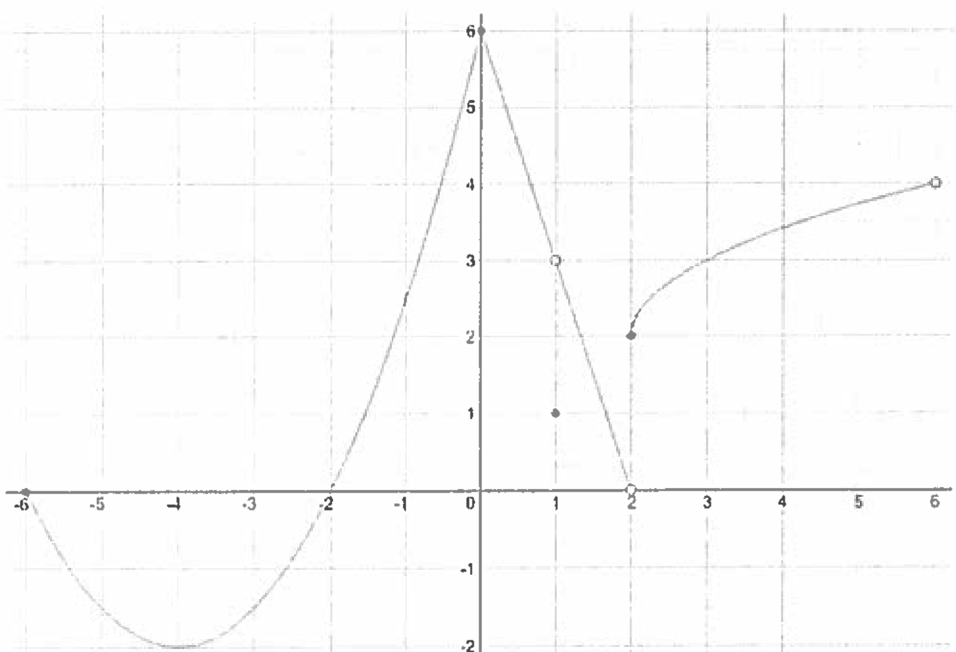
E) $\lim_{x \to 1} f(x)$ 3

F) $f(1)$ 1

G) $f(2)$ 2

H) $\lim_{x \to 2} f(x)$

DNE
LHL≠RHL



Properties of Limits

Let b and c be real numbers and let f and g be functions with the following limits:

$$\lim_{x\to c} f(x) = L \text{ and } \lim_{x\to c} g(x) = K$$

Rule	**Rule**
$\lim_{x\to c} b = b$	$\lim_{x\to c}[b \cdot g(x)] = bL$
$\lim_{x\to c} x = c$	$\lim_{x\to c}[f(x)]^{\frac{r}{s}} = L^{\frac{r}{s}}$ provided r and s are integers and $s \neq 0$.
$\lim_{x\to c}[f(x) \pm g(x)] = L \pm K$	$\lim_{x\to c}\left[\frac{f(x)}{g(x)}\right] = \frac{L}{K}$ provided $K \neq 0$.
$\lim_{x\to c}[f(x) \cdot g(x)] = L \cdot K$	

Example 1: Use the given information to evaluate the limits: $\lim_{x\to c} f(x) = 2$ and $\lim_{x\to c} g(x) = 3$

a) $\lim_{x\to c}[5g(x)]$ $5 \cdot 3 = 15$

b) $\lim_{x\to c}[f(x) + g(x)]$ $2 + 3 = 5$

c) $\lim_{x\to c}[f(x)g(x)]$ $2 \cdot 3 = 6$

d) $\lim_{x\to c}\frac{f(x)}{g(x)}$ $\frac{2}{3}$

Finding Limits by Direct Substitution

When tasked with evaluating a limit, the first thing you should always try is direct substitution. Direct substitution means that you plug-in the value that x approaches and determine what the expression evaluates to. As long as the expression does not evaluate to an undefined value, direct substitution will work.

Example 2: Evaluate the following limits.

a) $\lim_{x\to 1}(-x^2 + 1)$

$-1^2 + 1$

$= -1 + 1 = 0$

b) $\lim_{x\to 3}\frac{\sqrt{x+1}}{x-4}$ $\frac{\sqrt{3+1}}{3-4} = \frac{\sqrt{4}}{-1} = \frac{2}{-1} = -2$

c) $\lim_{h\to 0}(3h^2 + 2h)$

$3(0)^2 + 2(0) = 0$

d) $\lim_{h\to 0}(3x^2 - 2xh + 5h)$

$\cancel{3 \cdot 0^2 - 2 \cdot 0 \cdot h + 5h}$ $3x^2 - 2 \cdot 0 \cdot x + 5 \cdot 0$

$\cancel{= 5h}$ $= 3x^2$