$$\frac{f^n(2)(x-2)^n}{n!} = \frac{(n+1)!}{3^n}\frac{(x-2)^n}{n!} \to \frac{(x-2)^n(n+1)}{3^n}$$

BC Calculus Lagrange Error Bound **B** **Name:** ___________________________________

1. The Taylor series about $x = 2$ for a certain function f converges to $f(x)$ for all x in the interval of convergence. The nth derivative of f at $x = 2$ is given by

$$f^{(n)}(2) = \frac{(n+1)!}{3^n} \text{ and } f(2) = 1$$

a) Write the first four terms and the general term of the Taylor series for f about $x = 2$.

$$1 + \frac{2}{3}(x-2) + \frac{3}{9}(x-2)^2 + \frac{4}{27}(x-2)^3 + \dots + \frac{(x-2)^n(n+1)}{3^n}$$

b) Find the radius of convergence for the Taylor series for f about $x = 2$.

$$\lim_{n\to\infty}\left|\frac{(x-2)^{n+1}(n+2)}{3^{n+1}}\cdot\frac{3^n}{(x-2)^n(n+1)}\right| \to \lim_{n\to\infty}\left|\frac{(x-2)(n+2)}{3(n+1)}\right| = \left|\frac{x-2}{3}\right|$$

R.O.C $=3$

$$\left|\frac{x-2}{3}\right| < 1 \to |x-2| < 3$$

c) Let g be a function with $g(2) = 3$ and $g'(x) = f(x)$ for all x. Write the first four terms and the general term for the Taylor series for g about $x = 2$.

$$3 + (x-2) + \frac{1}{3}(x-2)^2 + \frac{1}{9}(x-2)^3 + \dots + \frac{\cancel{(n+1)}(x-2)^{n+1}}{\cancel{(n+1)}3^n}$$

2. Let f be the function defined by $f(x) = \sqrt{x}$.

a) Find the second-degree Taylor polynomial about $x = 4$ for the function f.

$f(x) = \sqrt{x} \to$ at $x=4 \to 2$

$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} \to$ at $x=4 \to \frac{1}{4}$

$f''(x) = -\frac{1}{4}x^{-\frac{3}{2}} \to$ at $x=4 \to -\frac{1}{32}$

$$P_2(x) = 2 + \frac{1}{4}\cdot(x-4) - \frac{1}{32}\frac{(x-4)^2}{2!}$$

b) Use your answer in part a to estimate the value of $f(4.2)$.

$$f(4.2) \approx P_2(4.2) = 2 + \frac{1}{4}(.2) - \frac{1}{64}(.2)^2 = 2 + \frac{1}{20} - \frac{1}{1600} = \frac{3200+80-1}{1600} = \frac{3279}{1600}$$

c) Find a bound on the error for the approximation in part b.

$$|P_2(4.2) - f(4.2)| \le \left|\frac{\frac{3}{256}(.2)^3}{3!}\right|$$

*we need

$f'''(x) \to \frac{3}{8}x^{-\frac{5}{2}} \to$ at $4 \to \frac{3}{8\cdot 32} \to \frac{3}{256}$

3. The Taylor series for a function f about $x = 1$ is given by $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{n}(x-1)^n$ and converges to $f(x)$ for $|x-1| < R$, where R is the radius of convergence of the Taylor series.

a) Find the value of R. $R = 1/2 \rightarrow$ b/c you end up at $|2(x-1)| < 1$

$$|x-1| < \frac{1}{2}$$

$\downarrow f$

b) Find the first three nonzero terms and the general term of the Taylor series for f', the derivative of f, about $x = 1$. Let's find some terms of f and derive them ($n=1$ to start)

$$f(1) + 2(x-1) - 2(x-1)^2 + \frac{8}{3}(x-1)^3 - 4(x-1)^4 + \cdots$$

so $f' \rightarrow 0 + 2 - 4(x-1) + 8(x-1)^2 - 16(x-1)^3 \;\; + \;\; + (-1)^{n+1}(x-1)^{n-1}\,2^n$

↳ Jus to see the pattern

c) The Taylor series for f' about $x = 1$, found in part b, is a geometric series. Find the function f' to which the series converges for $|x-1| < R$. Use the function to determine f for $|x-1| < R$.

$$S = \frac{a_1}{1-r} = \frac{2}{1-(-2(x-1))} = \frac{2}{1-(-2x+2)} = \frac{2}{1+2x-2} = \frac{2}{2x-1} \rightarrow \text{this is } f' \text{ so } f = \int_1^x \frac{2}{2x-1}$$

$$f = \frac{2\ln|2x-1|}{2}\Big|_1^x = \ln|2x-1| - \ln|2-1| \;\; \boxed{= \ln|2x-1|}$$

4. The function f is defined by the power series

$$f(x) = 1 + (x+1) + (x+1)^2 + \cdots + (x+1)^n + \cdots = \sum_{n=0}^{\infty}(x+1)^n$$

for all real numbers x for which the series converges.

Geo series $S = \frac{a_1}{1-r}$ / $S = \frac{1}{1-(x+1)}$

ratio

$S = f(x) = -\frac{1}{x}$

a) Find the interval of convergence of the power series for f.

$|x+1| < 1 \rightarrow$ radius

center -1

IOC

$-2 < x < 0$

Check.
$x=-2$ diverging
$x=0$ diverge

b) Let g be the function defined by $g(x) = \int_{-1}^{x} f(t)\,dt$. Find the value of $g\left(-\frac{1}{2}\right)$, if it exists, or explain why $g\left(-\frac{1}{2}\right)$ cannot be determined.

$$g(x) = \int_{-1}^{x} -\frac{1}{t}\,dt \rightarrow -\ln|t|\Big|_{-1}^{x} = -\ln|x| + \ln|-1| = -\ln|x| + 0$$

$$g\left(-\tfrac{1}{2}\right) = -\ln\left|-\tfrac{1}{2}\right| = -\ln\left(\tfrac{1}{2}\right) = -(\ln 1 - \ln 2) = -(-\ln 2) = \ln 2$$

c) Let h be the function defined by $f(x^2-1)$. Find the first three nonzero terms and the general term of the Taylor series for h about $x = 0$, and find the value of $h\left(\frac{1}{2}\right)$.

$$h(x) = 1 + (x^2-1+1) + (x^2-1+1)^2 + (x^2-1+1)^3 + \cdots + (x^2-1+1)^n$$

$$h(x) = 1 + x^2 + x^4 + x^6 + \cdots + x^{2n}$$

$$S = \frac{1}{1-x^2} \quad \frac{1}{1-\left(\frac{1}{2}\right)^2} = \frac{1}{1-\frac{1}{4}} = \frac{1}{3/4} = 4/3$$