BC Calculus Lagrange Error Bound Notesheet **Name: ______________________________**

When a Taylor polynomial is used to approximate a function, we need a way to see how accurately the polynomial approximates the function. How much the approximation is off by is referred to as the remainder, or the error of the Taylor polynomial. In general,

$$function = polynomial\ approximation + remainder$$

$$remainder = function - \ polynomial\ approximation$$

$$f(x) = P_n(x) + R_n(x) \text{ so } R_n(x) = f(x) - P_n(x)$$

To find the maximum amount of error, we will use the first discarded term of the approximation, much like we did when finding the error of an alternating series. There are two typical forms for this in AP Calculus. They essentially mean the same thing, they are just expressed a little differently.

Taylor's Inequality

If $|f^{(n+1)}(x)| \le M$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$R_n(x) = |f(x) - P_n(x)| \le \frac{M}{(n+1)!}|x-c|^{n+1}$$

Lagrange Error Bound (or Lagrange Remainder)

If a function f is differentiable through order $n+1$ in an interval containing c (the center), then for each x in the interval, there exists a number z between x and c such that

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + R_n(x)$$

Where the remainder $R_n(x)$ is given by $R_n(x) \le \left|\frac{|f^{(n+1)}(z)|}{(n+1)!}(x-c)^{n+1}\right|$, the Lagrange Error Bound

The key is to maximize the $|f^{(n+1)}(z)|$ (or M) and the $(x-c)^{n+1}$ pieces over the given interval. Sometimes you will be given $|f^{(n+1)}(z)|$ or M, sometimes you will be given an interval. If you are given neither, assume that the interval is from the center to the x-value you are using in your approximation.

Example 1 Let f be a function with 5 derivatives on the interval $[2, 3]$. Assume that $|f^{(5)}(x)| < 0.2$ for all x in the interval $[2, 3]$ and that a fourth-degree Taylor polynomial for f at $c = 2$ is used to estimate $f(3)$.

a) How accurate is the approximation? Give three decimal places.

$|P_4(x) - f(x)| \le \frac{|f^5(z)|(x-2)^5}{5!} \rightarrow$ Error $\le \frac{(.2)(3-2)^5}{5!} \rightarrow \frac{1}{5\,5!} \rightarrow \frac{1}{600}$

b) Suppose that $\boxed{P_4(3) = 1.763.}$ Use your answer in part a to find the interval in which $f(3)$ must reside.

the 4th degree Taylor approx at x=3 is 1763

$1763 - \frac{1}{600} < f(3) < 1.763 + \frac{1}{600}$

c) Could $f(3) = 1.778$? Why or why not? Could $f(3) = 1.764$? Why or why not?

* upper bound from (c) was $1764\bar{6}$ $\leftarrow$ $f(3) \neq 1.778 \rightarrow$ NO

$f(3) = 1764$

yes, possible

Example 2 The function f has derivatives of all orders for all real numbers x. Assume that $f(2) = 6$, $f'(2) = 4$, $f''(2) = -7$, and $f'''(2) = 8$.

a) Write the third-degree Taylor polynomial for f about $x = 2$, and use it to approximate $f(2.3)$.

$$P_3(x) = 6 + 4(x-2) - \frac{7(x-2)^2}{2!} + \frac{8(x-2)^3}{3!}$$

$$f(2.3) \approx 6 + 4(.3) - \frac{7(.3)^2}{2} + \frac{8(.3)^3}{6} = 6.921$$

b) The fourth derivative of f satisfies the inequality $|f^{(4)}(x)| \le 9$ for all x in the closed interval $[2, 2.3]$. Use the Lagrange error bound on the approximation $f(2.3)$ found in part a to find an interval $[a, b]$ such that $a \le f(2.3) \le b$.

$$\text{Error} \le \frac{9(.3)^4}{4!}$$

$$6.921 - .00304 \le f(2.3) \le 6.921 + .00304$$

$$6.918 \le f(2.3) \le 6.924$$

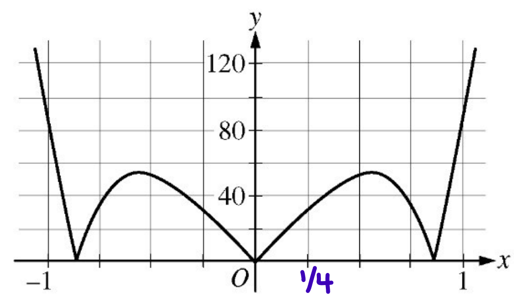
c) Based on the information above, could $f(2.3)$ equal 6.992? Explain why or why not.

$f(2.3) \neq 6.992 \rightarrow$ too large for the interval

Maclaurin series

Example 3 Let $f(x) = \sin(x^2) + \cos x$. The graph of $y = |f^{(5)}(x)|$ is shown below.

a) Write the first four nonzero terms of the Taylor series for $\sin x$ about $x = 0$, and write the first four nonzero terms of the Taylor series for $\sin(x^2)$ about $x = 0$.



Graph of $y = |f^{(5)}(x)|$

$$\sin(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

$$\sin(x^2) \approx x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!}$$

b) Write the first four nonzero terms of the Taylor series for $\cos x$ about $x = 0$. Use this series and the series for $\sin(x^2)$, found in part a, to write the first four nonzero terms of the Taylor series for f

$$\cos(x) \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

$$f(x) \approx 1 + \left(x^2 - \frac{x^2}{2!}\right) + \frac{x^4}{4!} + \left(-\frac{x^6}{3!} - \frac{x^6}{6!}\right) = 1 + \frac{x^2}{2} + \frac{x^4}{24} - \frac{121x^6}{720}$$

c) Find the value of $f^{(6)}(0)$.

$$\frac{f^6(0)\,x^6}{6!} = \frac{-121}{720}x^6 \qquad f^6(0) = -121$$

d) Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$. Using the information from the graph of $y = |f^{(5)}(x)|$ shown above, show that $\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| < \frac{1}{3000}$.

$$\left|P_4\left(\tfrac{1}{4}\right) - f\left(\tfrac{1}{4}\right)\right| \le \frac{40\left(\frac{1}{4}\right)^5}{5!} = \frac{1}{3072} \qquad \frac{1}{3072} < \frac{1}{3000}$$

$$\frac{\cancel{4}\,\cancel{10}}{4^5 \cdot \cancel{5} \cdot \cancel{4} \cdot 3 \cdot \cancel{2} \cdot 1} = \frac{1}{1024 \cdot 3}$$

general term $\frac{f^n(3)(x-3)^n}{n!} = \frac{(-1)^n n!}{5^n(n+3)} \frac{(x-3)^n}{n!} = \frac{(-1)^n (x-3)^n}{5^n(n+3)}$

Example 4 The Taylor series about $x = 3$ for a certain function f converges to $f(x)$ for all x in the interval of convergence. The nth derivative of f at $x = 3$ is given by

$$\boxed{f^{(n)}(3) = \frac{(-1)^n n!}{5^n(n+3)}} \text{ and } f(3) = \frac{1}{3}$$

a) Write the fourth-degree Taylor polynomial for f about $x = 3$.

$$P_4(x) = \frac{1}{3} - \frac{(x-3)}{20} + \frac{(x-3)^2}{125} - \frac{(x-3)^3}{750} + \frac{(x-3)^4}{4375}$$

b) Find the radius of convergence of the Taylor series for f about $x = 3$.

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x-3)^{n+1}}{5^{n+1}(n+4)} \cdot \frac{5^n(n+3)}{(-1)^n(x-3)^n}\right| \to \lim_{n\to\infty}\left|\frac{(-1)(x-3)(n+3)}{5\quad(n+4)}\right| = \left|\frac{x-3}{5}\right| \quad |x-3|<5$$

$$\left|\frac{x-3}{5}\right|<1 \qquad \boxed{\text{Radius} = 5}$$

c) Show that the third-degree Taylor polynomial approximate $f(4)$ with an error less than $\frac{1}{4000}$.

$$|P_3(4) - f(4)| \le \frac{|f^4(3)|(4-3)^4}{4!} \qquad \frac{4!}{5^4 \cdot 7 \cdot 4!} = \frac{1}{4375} \qquad \frac{1}{4375} < \frac{1}{4000}$$

Example 5 Let h be a function having derivatives of all orders for $x > 0$. Selected values of h and its first four derivatives are indicated in the table to the right. The function h and these four derivatives are increasing over the interval $1 \le x \le 3$.

x	$h(x)$	$h'(x)$	$h''(x)$	$h'''(x)$	$h^{(4)}(x)$
1	11	30	42	99	18
2	80	128	$\frac{488}{3}$	$\frac{448}{3}$	$\frac{584}{9}$
3	317	$\frac{753}{2}$	$\frac{1383}{4}$	$\frac{3483}{16}$	$\frac{1125}{16}$

a) (1) Write the first-degree Taylor polynomial for h about $x = 2$ and (2) use it to approximate $h(1.9)$. (3) Is this approximation greater than or less than $h(1.9)$? Explain your reasoning.

(1) $P_1(x) = 80 + 128(x-2)$

(2) $h(1.9) \approx 80 + 128(-.1)$
$= 80 - 12.8 = 67.2$

(3) $h''(2) = \frac{488}{3}$ which is +, so $h(x)$ at $x=2$ is concave up, so 67.2 is an underestimate

b) (1) Write the third-degree Taylor polynomial for h about $x = 2$ and (2) use it to approximate $h(1.9)$.

$$P_3(x) = 80 + 128(x-2) + \frac{488}{3}\frac{(x-2)^2}{2!} + \frac{448}{3}\frac{(x-2)^3}{3!}$$

(2) $h(1.9) \approx 67.988$
(Just plug in 1.9 for x)

c) Use the Lagrange error bound to show that the third-degree Taylor polynomial for h about $x = 2$ approximates $h(1.9)$ with error less than $3 \cdot 10^{-4}$.

$$|P_3(1.9) - h(1.9)| \le \frac{\left|\frac{584}{9}\right|(-.1)^4}{4!} \longrightarrow 2.703 \times 10^{-4} < 3\times10^{-4}$$