When Limits Do Not Exist

There are times when a limit fails to exist, meaning that we are unable to find a value for the limit at the given point. When this happens, we say that the limit **does not exist**, abbreviating this statement as **DNE**. There are three types of common function behavior that lead to a limit that does not exist.

Common Types of Behavior Associated with a Limit that Does Not Exist

1. $f(x)$ approaches a different number from the right side of c than it approaches from the left side.
2. $f(x)$ increases or decreases without bound as x approaches c.
3. $f(x)$ oscillates between two fixed values as x approaches c.

Note: We will be more specific about Cause Number 2 when we explore infinite limits.

Example 4: Investigate the existence of the following limit using a graph and table.

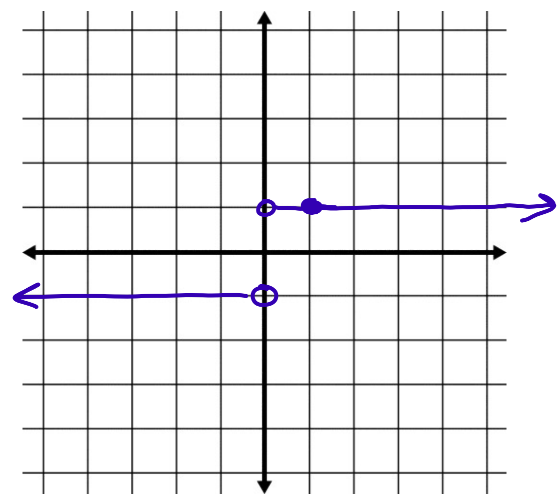
excluded value $x=0$

$$\lim_{x \to 0} \frac{|x|}{x}$$

$$f(x) = \begin{cases} 1, & x > 0 \\ -1, & x \quad 0 \end{cases}$$

$$\frac{|x|}{x} = \frac{x}{x} \text{ or } \frac{-x}{x}$$

$$= 1 \text{ or } -1$$

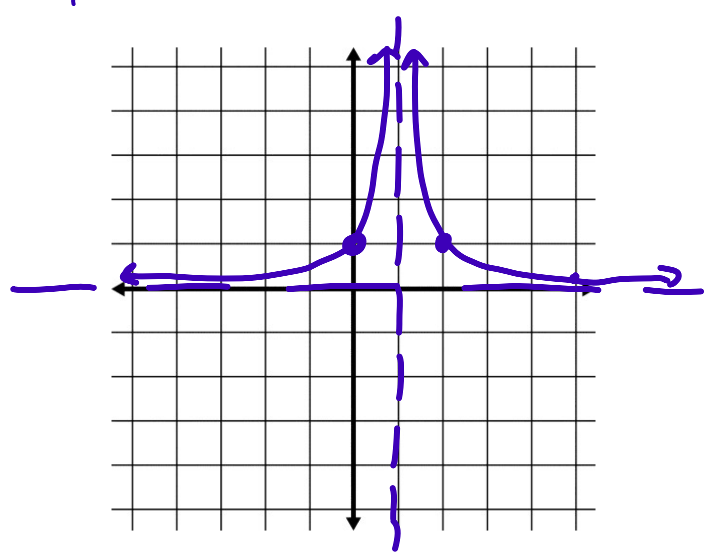


x	-0.5	-0.25	-0.1	-.01	-.001	0	.001	.01	.1	.25	.5
$f(x)$	-1	-1	-1	-1	-1	undefined	1	1	1	1	1

Example 5: Investigate the existence of the following limit using a graph and table.

$$\lim_{x \to 1} \frac{1}{(x-1)^2} = \infty$$

excluded value $x=1$



x	0.5	0.75	0.9	0.99	0.999	1	1.001	1.01	1.1	1.25	1.5
$f(x)$	4	16	100	10,000	1,000,000	undefined	1,000,000	10,000	100	16	4

Example 6: Use a table to investigate $\lim_{x\to 0^+} \sin\left(\frac{1}{x}\right)$

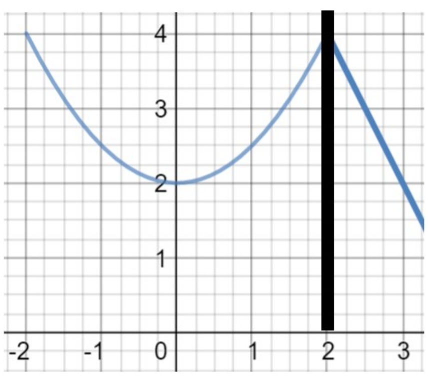
x	$\frac{2}{\pi}$	$\frac{2}{3\pi}$	$\frac{2}{5\pi}$	$\frac{2}{7\pi}$	$\frac{2}{9\pi}$	$\frac{2}{11\pi}$	$\frac{2}{13\pi}$	As $x \to 0^+$
$f(x)$	1	−1	1	−1	1	−1	1	Does not exist ⇒ oscillating

Finding Limits from Graphs

Hopefully you are getting comfortable with this new idea of a limit value and how it is different from a function value. When trying to find limits and function values from graphs, refer to the following visual aids:

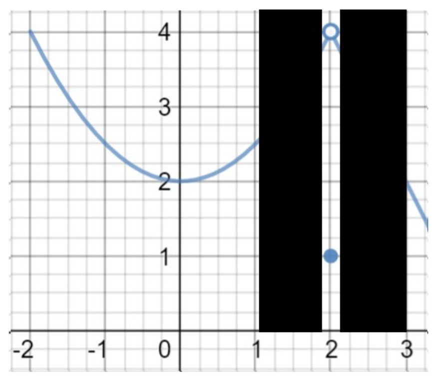
When looking for a limit value at $x = c$, imagine that you have got a thick vertical line covering up $x = c$ with only the graph showing on either side of $x = c$. You are now looking to see what y-value the graph is approaching on either side of $x = c$. If the graphs appear to be approaching the same y-value, the limit exists and is that y-value. Otherwise, the limit does not exist.

When looking for a function value at $x = c$, imagine that you have got shudders on either side of $x = c$ with only a vertical sliver at $x = c$ visible between them. You are now looking for the dot or the piece of graph that exists in that narrow sliver. If it exists, the y-value of the dot is the function value $f(c)$. Otherwise, the function value is undefined.



$\lim_{x\to 2} f(x) = 4$

$f(2) = 1$



Example 7: Use the graph to evaluate the following limits

A) $\lim_{x\to 0} f(x) = 6$

B) $\lim_{x\to 6^-} f(x) = 4$

C) $\lim_{x\to 2^-} f(x) = 0$

D) $\lim_{x\to -6} f(x)$ Does not exist (not enough info)

E) $\lim_{x\to 1} f(x) = 3$

F) $f(1) = 1$

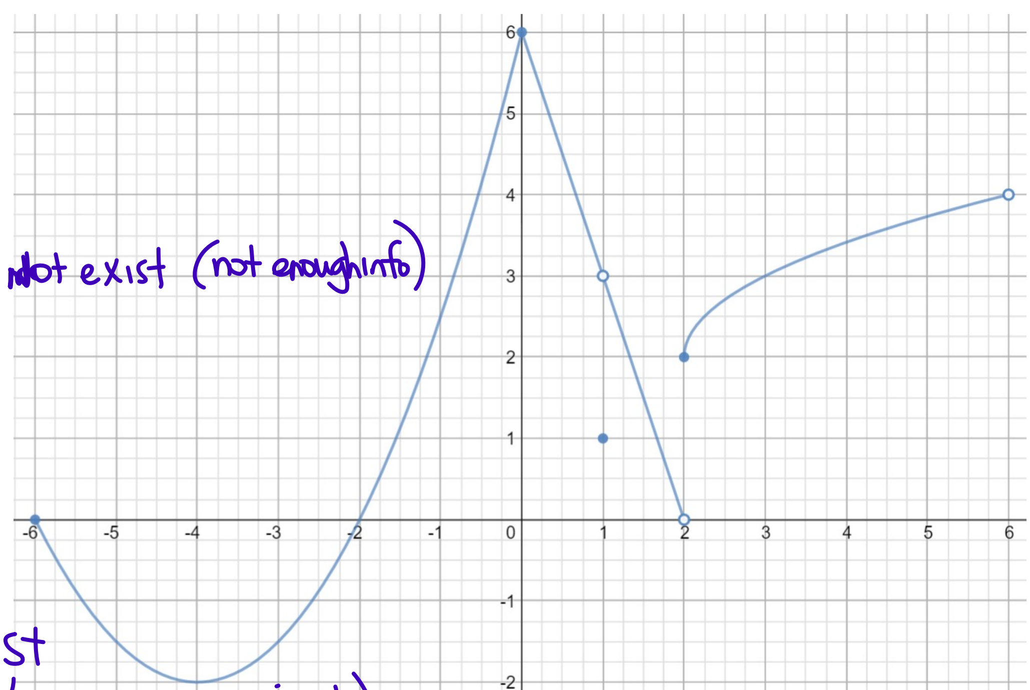
G) $f(2) = 2$

H) $\lim_{x\to 2} f(x)$

Does not exist

(Left side Limit ≠ Right side Limit)

0 ≠ 2



Properties of Limits

Let b and c be real numbers and let f and g be functions with the following limits:

$$\lim_{x\to c} f(x) = L \text{ and } \lim_{x\to c} g(x) = K$$

Rule	**Rule**
$\lim_{x\to c} b = b$	$\lim_{x\to c}[b\cdot g(x)] = bK$
$\lim_{x\to c} x = c$	$\lim_{x\to c}[f(x)]^{\frac{r}{s}} = L^{\frac{r}{s}}$ provided r and s are integers and $s \neq 0$.
$\lim_{x\to c}[f(x) \pm g(x)] = L \pm K$	$\lim_{x\to c}\left[\frac{f(x)}{g(x)}\right] = \frac{L}{K}$ provided $K \neq 0$.
$\lim_{x\to c}[f(x)\cdot g(x)] = L\cdot K$	

Example 1: Use the given information to evaluate the limits: $\boxed{\lim_{x\to c} f(x) = 2}$ and $\boxed{\lim_{x\to c} g(x) = 3}$

a) $\lim_{x\to c}[5g(x)]$

$5\lim_{x\to c} g(x) = 5\ 3 = 15$

b) $\lim_{x\to c}[f(x) + g(x)]$

$\lim_{x\to c} f(x) + \lim_{x\to c} g(x) = 2 + 3 = 5$

c) $\lim_{x\to c}[f(x)g(x)]$

$\lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x) = 2\ 3 = 6$

d) $\lim_{x\to c}\frac{f(x)}{g(x)}$ $\Rightarrow \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \dfrac{2}{3}$

Finding Limits by Direct Substitution

When tasked with evaluating a limit, the first thing you should always try is direct substitution. Direct substitution means that you plug-in the value that x approaches and determine what the expression evaluates to. As long as the expression does not evaluate to an undefined value, direct substitution will work.

Example 2: Evaluate the following limits.

$-1\cdot x^2$

a) $\lim_{x\to 1}(-x^2 + 1) = -1(1)^2 + 1 = 0$

b) $\lim_{x\to 3}\frac{\sqrt{x+1}}{x-4} = \frac{\sqrt{3+1}}{3-4} = \frac{2}{-1} = -2$

c) $\lim_{h\to 0}(3h^2 + 2h) = 3(0)^2 + 2(0)$

$= 0$

d) $\lim_{h\to 0}(3x^2 - 2xh + 5h)$

$= 3x^2 - 2x(0) + 5(0)$

$= 3x^2$