AB Calculus: Finding Limits Algebraically **Name:** ______________________________

When evaluating a limit, the technique you always want to try first is direct substitution. If a limit cannot be found using direct substitution, then we will use other techniques to try and evaluate the limit. Keep in mind that some functions do not have limits.

> If direct substitution yields $\frac{0}{0}$, otherwise known as **indeterminate form**, then you cannot determine the limit in its current form. Encountering this form means you should try another technique. One way to deal with limits in this form is to use algebraic techniques like **factoring**, **simplifying**, and **rationalizing the numerator**. (or denominator)

Example 3: Evaluate the following limits.

a) $\lim_{x\to -1} \frac{2x^2 - x - 3}{x+1} = \lim_{x\to -1} \frac{(2x-3)\cancel{(x+1)}}{\cancel{x+1}}$ $\quad 2(-1)-3 = -5$

$\frac{2(-1)^2-(-1)-3}{-1+1} = \frac{0}{0}$ Indeterminate

b) $\lim_{x\to 3} \frac{(\sqrt{x+1}-2)}{(x-3)} \cdot \frac{(\sqrt{x+1}+2)}{(\sqrt{x+1}+2)} \to \lim_{x\to 3} \frac{\cancel{(x+1)-4}^{\,x-3}}{\cancel{(x-3)}(\sqrt{x+1}+2)}$ $\quad \frac{1}{\sqrt{3+1}+2} = \frac{1}{4}$

$\frac{\sqrt{3+1}-2}{3-3} = \frac{0}{0}$ Indeterminate

c) $\lim_{x\to 0} \frac{\frac{1}{x+4} - \frac{1}{4}}{x}$ $\quad \lim_{x\to 0} \frac{\frac{4}{4}\cdot\frac{1}{x+4} - \frac{1}{4}\cdot\frac{x+4}{x+4}}{x}$ $\quad \lim_{x\to 0} \frac{\frac{4}{4(x+4)} - \frac{1(x+4)}{4(x+4)}}{\frac{x}{1}} \cdot \frac{1}{x}$

$\frac{\frac{1}{0+4} - \frac{1}{4}}{0} = \frac{0}{0}$ Indeterminate

$\lim_{x\to 0} \frac{\cancel{4} - \cancel{x}^{1} - \cancel{4}}{4(x+4)\cancel{x}}$ $\quad \frac{-1}{4(0+4)} = \frac{-1}{16}$

d) $\lim_{x\to 2} \frac{3x^2 + 5x - 2}{x - 2}$ $\quad$ Does not exist

$\frac{3(2)^2+5(2)-2}{2-2} = \frac{20}{0}$

$x\to 2^-$ yields $-\infty$

$x\to 2^+$ yields $+\infty$

If direct substitution yields to a fraction with 0 in the denominator and something other than 0 in the numerator, then either you have a limit that increases or decreases without bound (∞ or $-\infty$), or the limit does not exist.

$\frac{\sin(0)}{0} = \frac{0}{0}$

Example 4: Investigate $\lim_{x\to 0} \frac{\sin x}{x}$ by looking at the graph and making a table.

x	-0.1	-.01	-.001	0	.001	.01	.1
$f(x)$	.9983	.999	99999	undefined	99999	999	9983

Based on your investigation, what do you think the value of the limit is?

$$\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1}$$

You must understand that while using a graph and/or table allows us to estimate the value of a limit, we have not proved it is the value until we algebraically confirm the limit is what we think it is. The proof of the limit in example 4 is a little more complicated, but for now, you will want to memorize this limit.

Example 5: Evaluate the following limits.

a) $\lim_{k\to 0}\frac{\sin k}{k} = 1$

b) $\lim_{x\to 0}\frac{\sin 5x}{4x} \cdot \frac{5x}{5x}$

$\lim_{x\to 0}\frac{\cancel{\sin 5x}}{\cancel{5x}}^{1} \cdot \frac{5\cancel{x}}{4\cancel{x}} = \frac{5}{4}$

c) $\lim_{x\to 0}\frac{\sin x}{5x^2+x}$ $\lim_{x\to 0}\frac{\sin x}{x(5x+1)}$

$\lim_{x\to 0}\frac{\cancel{\sin x}}{\cancel{x}}^{1}\ \frac{1}{5x+1} = 1$

The Sandwich Theorem (aka The Squeeze Theorem)

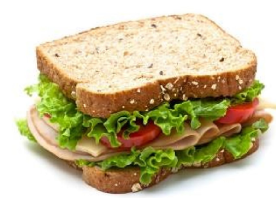
The Sandwich Theorem

If $g(x) \le f(x) \le h(x)$ for all $x \ne c$ in some interval about c, and

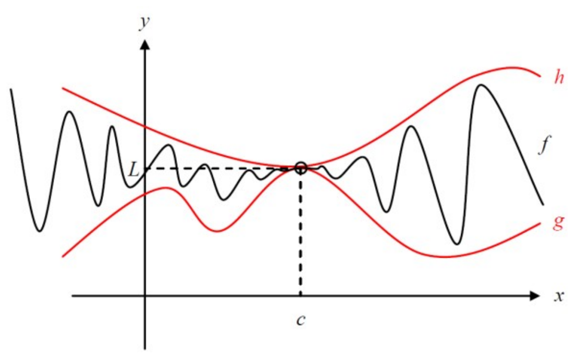
$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L,$$

then

$$\lim_{x\to c} f(x) = L$$



In other words, if we "sandwich" the function $f(x)$ between two other functions $g(x)$ and $h(x)$ that both have the same limit as x approaches c, then $f(x)$ is forced to have the same limit too.



Example 6: Use the Sandwich Theorem to evaluate the following limits.

a) If $5 - 3x - x^2 \le g(x) \le x + 9$ find $\lim_{x\to -2} g(x)$

$\lim_{x\to -2} 5-3x-x^2 = 5-3(-2)-(-2)^2 = 7$

$\lim_{x\to -2} x+9 = 7$ therefore by squeeze theorem $\lim_{x\to -2} g(x) = 7$

b) $\lim_{x\to 0} x^2 \cos\left(\frac{1}{x}\right)$

$-x^2 \le x^2 \cos\left(\frac{1}{x}\right) \le x^2$

$\lim_{x\to 0} -x^2 = 0$ $\lim_{x\to 0} x^2 = 0$ therefore by squeeze theorem $\lim_{x\to 0} x^2\cos\left(\frac{1}{x}\right) = 0$