AB Calculus: Limits Involving Infinity

We are going to look at two kinds of limits involving infinity. The first type is determining what happens to a function as x approaches infinity in either the positive or negative direction ($x \to \pm\infty$). The second type is functions whose limit approaches infinity in either the positive and negative direction as x approaches a given value.

The first type: Limits as $x \to \pm\infty$

Example 1: Use your calculator or Desmos to investigate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ for the following functions:

a) $f(x) = \frac{1}{x}$ $\quad \lim_{x\to\infty} \frac{1}{x} = 0 \quad \lim_{x\to-\infty} \frac{1}{x} = 0$

b) $f(x) = \frac{2x-1}{x+3}$ $\quad \lim_{x\to\infty} f(x) = 2 \quad \lim_{x\to-\infty} f(x) = 2$

Definition: Horizontal Asymptote

The line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = b \text{ or } \lim_{x\to-\infty} f(x) = b$$

The horizontal asymptotes of rational functions (the quotient of two polynomials like in example 1) will be the same in both the positive and negative directions. When dealing with rational functions, you can find horizontal asymptotes using the same rules you learned in previous math classes.

Using Rules for Finding Horizontal Asymptotes to find Infinite Limits

Relationship	Asymptote	Example
Degree of Numerator is Larger	No H A but instead slant/oblique	$\frac{5x^6+5x^4+8x}{10x^5-10x^3-6}$
Degree of Denominator is Larger	$y=0$	$\frac{5x^2-3x^3+6}{x^4}$
Degrees are the Same	y = ratio of coefficients $y=-\frac{3}{7}$ ←	$\frac{3x^2-5x+6}{-7x^2+6x-10}$

Another technique that can be used to find the horizontal asymptotes of a function is to use an end behavior model (EBM). An EBM is a function that is simpler than the original, but behaves in the exact same way as the original as x gets really big in either the positive or negative direction.

Finding Horizontal Asymptotes Using End Behavior Models

End Behavior Model for Rational Functions

For a rational function $\frac{ax^m + \dots}{bx^n + \dots}$ where m is the degree of the numerator and n is the degree of the denominator, the end behavior model, or the function that the original behaves like when x gets sufficiently large in the negative or positive direction, can be written as

$$\frac{ax^m}{bx^n} \text{ or } \frac{a}{b}x^{m-n}$$

what will the equation act like as you approach ∞ (right) or $-\infty$ (left)

Example 2: Find the end behavior model (EBM) for the following functions.

the fact that it says "lim x→∞" tells me to look at EBM

a) $\lim_{x\to\infty} \frac{3x^4 - 2x^3 + 3x^2 - 5x + 6}{5x^4}$ $\quad \frac{3x^4}{5x^4} = \frac{3}{5}$

EBM $y = \frac{3}{5}$

b) $\lim_{x\to\infty} \frac{2x^5 + x^4 - x^2 + 1}{3x^2 - 5x + 7}$ $\quad \frac{2x^5}{3x^2} = \frac{2}{3}x^3$

c) $\lim_{x\to\infty} \frac{2x^3 - x^2 + x - 1}{5x^2 + x^3 + x - 5}$ $\quad \frac{2x^3}{x^3} = 2$

d) $\lim_{x\to\infty} \frac{x^2 + x - 1}{x^4 - 3x^2 + 2x - 6}$ $\quad \frac{x^2}{x^4} = \frac{1}{x^2}$

(that means approach "0")

Example 3: For each of the following, find the end behavior model, evaluate the limit, and find any horizontal asymptotes.

a) $\lim_{x\to\infty} \frac{2x + 5}{3x^2 - 6x + 1}$ it will act like $\lim_{x\to\infty} \frac{2}{3x} = 0$ HA $y = 0$

EBM $\frac{2x}{3x^2} = \frac{2}{3x}$

b) $\lim_{x\to\infty} \frac{2x^2 - 3x + 5}{x^2 + 1}$ it will act like $\Rightarrow \lim_{x\to\infty} 2 = 2$ HA $y = 2$

EBM $\frac{2x^2}{x^2} = 2$

c) $\lim_{x\to-\infty} \frac{x^4 + x^3 + 9}{3x - 3}$ $\quad \lim_{x\to-\infty} \frac{1}{3}x^3 = -\infty$ HA None (slant/oblique)

EBM: $\frac{x^4}{3x} = \frac{1}{3}x^3$

Horizontal Asymptotes of Non-Rational Functions

In previous examples, the horizontal asymptotes in both the positive and negative direction were the same. Once again, this occurs whenever you have a rational function. This is not necessarily the case for other functions that are created from the quotient of two functions. There are even functions that have more than one horizontal asymptote! As a general rule for functions that are divided, if the denominator "grows" faster that the numerator in the given direction, the limit as x approaches infinity will be 0; if the numerator "grows" faster than the denominator, the limit will increase or decrease without bound; if they grow at the same rate, then the limit will be a constant value.

Growth Rate
Factorial
Exponential
Polynomial
Logarithmic
sin x/cos x
constants

Example 4: Evaluate the following limits.

a) $\lim_{x\to\infty} \frac{\sin x}{x}$ $\frac{\text{bounce back and forth between } 1 \text{ \& } -1}{\text{approaches } \infty}$

$= 0$

b) $\lim_{x\to-\infty} e^x - 2x$

(algebra?)

$e^{-\infty} - 2(-\infty)$

$\frac{1}{e^{\infty}} + 2\infty \to 0 + \infty \to \infty$

c) $\lim_{x\to\infty} \frac{5\boxed{+}2^x}{2\boxed{-}2^x}$ -1

$\frac{5+2^{\infty}}{2-2^{\infty}}$

d) $\lim_{x\to-\infty} \frac{5+2^x}{2-2^x}$

$\frac{5+2^{-\infty}}{2-2^{-\infty}} \to \frac{5+\frac{1}{2^{\infty}}}{2-\frac{1}{2^{\infty}}} \to \frac{5+0}{2-0}$

$= \frac{5}{2}$

The Second Type: Infinite Limits as $x \to a$

Example 5: Investigate the following limits.

a) $\lim_{x\to0^-} \frac{1}{x}$

b) $\lim_{x\to0^+} \frac{1}{x}$

Definition: Vertical Asymptote

The line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x\to a^-} f(x) = \pm\infty \text{ or } \lim_{x\to a^+} f(x) = \pm\infty$$

Note: This occurs whenever there is a value of x that gives you a 0 in the denominator but not the numerator. If the value of x gives you a 0 in both the numerator and the denominator, the graph has a hole.