Horizontal Asymptotes of Non-Rational Functions

In previous examples, the horizontal asymptotes in both the positive and negative direction were the same. Once again, this occurs whenever you have a rational function. This is not necessarily the case for other functions that are created from the quotient of two functions. There are even functions that have more than one horizontal asymptote! As a general rule for functions that are divided, if the denominator "grows" faster that the numerator in the given direction, the limit as x approaches infinity will be 0; if the numerator "grows" faster than the denominator, the limit will increase or decrease without bound; if they grow at the same rate, then the limit will be a constant value.

Growth Rate
Factorial
Exponential
Polynomial
Logarithmic
sin x/cos x
constants

Example 4: Evaluate the following limits.

a) $\lim\limits_{x\to\infty} \frac{\sin x}{x}$ → $\frac{\text{bounce back and forth between } 1, -1}{\text{approaches } \infty}$

$= 0$

b) $\lim\limits_{x\to-\infty} e^x - 2x$

(algebra?)

$e^{-\infty} - 2(-\infty)$

$\frac{1}{e^{\infty}} + 2\infty \to 0 + \infty \to \infty$

c) $\lim\limits_{x\to\infty} \frac{5\boxed{+}2^x}{2\boxed{-}2^x}$ -1

$\frac{5+2^{\infty}}{2-2^{\infty}}$

d) $\lim\limits_{x\to-\infty} \frac{5+2^x}{2-2^x}$

$\frac{5+2^{-\infty}}{2-2^{-\infty}} \to \frac{5+\frac{1}{2^{\infty}}}{2-\frac{1}{2^{\infty}}} \to \frac{5+0}{2-0}$

$= \frac{5}{2}$

The Second Type: Infinite Limits as $x \to a$

Example 5: Investigate the following limits.

a) $\lim\limits_{x\to 0^-} \frac{1}{x}$ $-\infty$

b) $\lim\limits_{x\to 0^+} \frac{1}{x}$ ∞

\# what is $\lim\limits_{x\to 0} \frac{1}{x}$? Does not exist

Definition: Vertical Asymptote

The line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x\to a^-} f(x) = \pm\infty \text{ or } \lim_{x\to a^+} f(x) = \pm\infty$$

Note: This occurs whenever there is a value of x that gives you a 0 in the denominator but not the numerator. If the value of x gives you a 0 in both the numerator and the denominator, the graph has a hole.

Factor the denominator

Example 6: Find the vertical asymptotes of each function, find the limit of the function as x approaches from the left and the right of each asymptote, then find the value of the limit at each asymptote.

a) $f(x) = \frac{x^2 - 1}{2x - 4} = \frac{(x-1)(x+1)}{2(x-2)}$

VA $x = 2$

$\lim_{x \to 2^-} f(x) \to -\infty$ $\lim_{x \to 2^+} f(x) \to \infty$

$\lim_{x \to 2} f(x)$ Does not exist

b) $f(x) = \frac{1 - x}{x^2 - 4x + 3} = \frac{-\cancel{(x-1)}}{(x-3)\cancel{(x-1)}} = \frac{-1}{x-3}$

VA $x = 3$

$\lim_{x \to 3^-} f(x) \to \infty$ $\lim_{x \to 3^+} f(x) \to -\infty$ $\lim_{x \to 3} f(x)$ DNE

c) $f(x) = \frac{1}{(x + 1)^2}$

$\lim_{x \to -1^-} f(x) \to \infty$ $\lim_{x \to -1^+} f(x) \to \infty$ $\lim_{x \to -1} f(x) \to \infty$

VA $x = -1$

In general, if the limit increases without bound from both the left and the right of the asymptote, the value of the limit is ∞. If the limit decreases without bound from both the left and the right, the limit is $-\infty$. If the limit from the left and right are not the same, we say the limit does not exist (DNE). Remember, a limit value of ∞ or $-\infty$ technically means the limit does not exist (it is one of the three common behaviors discussed previously that caused limits not to exist). However, we give the more specific answer of ∞ or $-\infty$ because it distinguishes between limits that behave the same way from both sides of the value, and limits that do not. From this day forward, DNE is not an acceptable answer for a limit value that should be ∞ or $-\infty$.

Graphs and Limits involving infinity

Example 8: Use the graph to evaluate the following limits

A $f(0) = -1$

B $\lim_{x \to -\infty} f(x) = -2$

C $\lim_{x \to \infty} f(x) = 0$

D $\lim_{x \to 2^+} f(x)$ ∞

E $\lim_{x \to 2^-} f(x)$ $-\infty$

F $f(2)$ undefined