AP Calculus Linearization and Differentials Homework Name: Key

1. If $f(x) = x^3 - 2x + 3$,

a) Find the linearization $L(x)$ of $f(x)$ centered at $x = 2$.

$f(a) = f(2) = 8-4+3 = 7$ $L(x) = 10(x-2)+7$

$f'(a) = 3a^2 - 2 \rightarrow f'(2) = 10$

b) Use $L(x)$ to approximate $f(2.1)$

$f(2.1) \approx L(2.1) = 10(2.1-2)+7 = 10(.1)+7 = 8$

c) Determine if the approximation is an underestimate or an overestimate.

$f''(x) = 6x$ $\overset{-}{\longleftarrow}\underset{0}{|}\overset{+\,ccu}{\underset{2}{|}\longrightarrow}$ so it's an underestimate

d) Use a calculator to determine the accuracy of the approximation you found in part b.

$|8.061-8| = .061$ Error $< 10^{-1}$

2. If $f(x) = \sqrt{x^2+9}$,

a) Find the linearization $L(x)$ of $f(x)$ centered at $x = -4$.

$f(-4) = \sqrt{25} = 5$ $f'(-4) = \frac{-4}{5}$ $L(x) - 5 = -\frac{4}{5}(x+4)$

$f'(x) = \frac{1}{2\sqrt{x^2+9}} \cdot 2x = \frac{x}{\sqrt{x^2+9}}$ $L(x) = -\frac{4}{5}(x+4)+5$

b) Use $L(x)$ to approximate $f(-3.9)$

$f(-3.9) \approx L(-3.9) = -\frac{4}{5}(-3.9+4)+5 = (-.8)(.1)+5 = -.08+5 = 4.92$

c) Use a calculator to determine the accuracy of the approximation you found in part b.

$|4.92036584 - 4.92|$ Error $< 10^{-3}$

3. Approximate the value of $\sqrt{101}$ by using a linearization of a nearby number.

$f(x) = \sqrt{x}$ $f(100) = 10$ $L(x) - 10 = \frac{1}{20}(x-100)$

$f'(x) = \frac{1}{2\sqrt{x}}$ $f'(100) = \frac{1}{20}$ $L(x) = \frac{1}{20}(x-100)+10$

$f(101) \approx L(101) = \frac{1}{20}(101-100)+10 = .05(1)+10 = 10.05$

4. For each of the following, find dy and evaluate dy for the given value of x and dx.

a) $y = x^3 - 3x$, $x = 2$, $dx = 0.05$

$\frac{dy}{dx} = 3x^2 - 3 \qquad dy = (3(2)^2 - 3).05 = .45$

b) $y = x^2 \ln x$, $x = 1$, $dx = 0.01$

$\frac{dy}{dx} = 2x \ln x + x^2 \cdot \frac{1}{x} \rightarrow dy = (2x \ln x + x)\,dx$

$dy = (2(1)\ln(1) + 1) \cdot .01$

$= .01$

5. Use differentials to estimate the change in the volume of a sphere $\left(V = \frac{4}{3}\pi r^3\right)$ when the radius changes from 10 cm to 10.05 cm.

$\frac{dV}{dr} = \frac{4}{3} \cdot 3 \cdot \pi r^2 \Rightarrow dV = 4\pi r^2 dr \qquad r = 10 \quad dr = .05$

$= 4\pi(10)^2(.05)$

$= 20\pi \text{ cm}^3$

6. Evaluate the following limits.

a) $\lim_{x \to 0^+} x \ln x \longrightarrow \lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x}} \rightarrow \frac{\infty}{\infty} \qquad \frac{\frac{1}{x}}{\frac{-1}{x^2}} = -x \rightarrow 0$

$0 \cdot \ln 0$

$0 \cdot \infty$

b) $\lim_{x \to \frac{\pi}{2}^+} \frac{1 + \sec x}{\tan x} \quad \frac{\infty}{\infty} \qquad \lim_{x \to \frac{\pi}{2}^+} \frac{\sec x \tan x}{\sec^2 x} \Rightarrow \frac{\tan x}{\sec x} \rightarrow \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x}} \rightarrow \sin x$

$\rightarrow \sin \frac{\pi}{2} = 1$

c) $\lim_{x \to 0} (e^x + x)^{\frac{1}{x}}$

$\frac{1}{x} \ln(e^x + x) \rightarrow \frac{\ln(e^x + x)}{x} \Rightarrow \frac{0}{0} \qquad \lim_{x \to 0} \frac{\frac{1}{e^x + x}(e^x + 1)}{1}$

$\lim_{x \to 0} \frac{e^x + 1}{e^x + x} \rightarrow \frac{2}{1} = 2$

undo ln $\quad e^2$