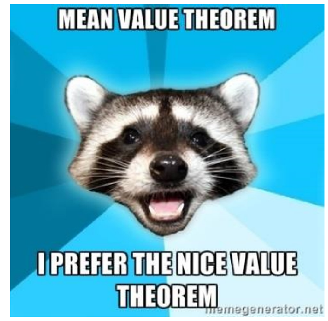
AB Calculus: The Mean Value Theorem **Name:** ______________________________

The Mean Value Theorem (MVT) is considered by some to be the most important topic in all of calculus other than limits. It is used to prove many of the theorems in calculus that we use in this course as well as further studies into calculus.



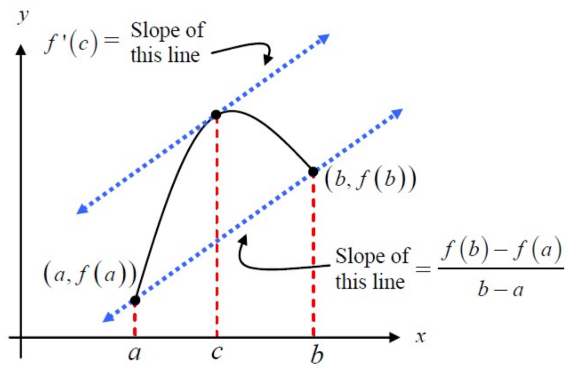
The Mean Value Theorem

If f is continuous over the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Some important notes regarding the Mean Value Theorem

- Just like the Intermediate Value Theorem, this is an existence theorem. The Mean Value Theorem does not tell you what the value of c is, nor does it tell you how many exist.
- The hypothesis of the Mean Value Theorem is highly important. If any part of the hypothesis does not hold, the theorem cannot be applied.
- Remember, c is an x-value.
- Basically, the Mean Value Theorem says that the average rate of change over the entire interval is equal to the instantaneous rate of change at some point inside the interval.



Example 1 Apply the Mean Value Theorem to the function on the indicated interval. In each case, make sure the hypothesis is true, then find all values of c inside the interval guaranteed by the MVT.

Differentiable on $(-1,1)$ ✓

a) $f(x) = x(x^2 - x - 2)$ over the interval $[-1, 1]$

$f(x) = x^3 - x^2 - 2x$

$f'(x) = 3x^2 - 2x - 2$

$f(1) = 1 - 1 - 2 = -2$

$f(-1) = -1 - 1 + 2 = 0$

$\frac{0 - -2}{-1 - 1} = \frac{2}{-2} = -1$

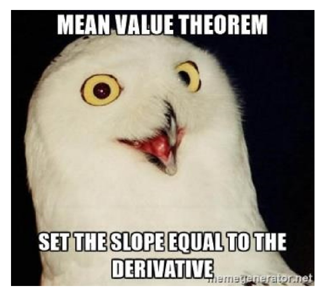
$3x^2 - 2x - 2 = -1$

$3x^2 - 2x - 1 = 0$

$(3x + 1)(x - 1) = 0$

$x = -\frac{1}{3}$ $\quad x = 1$

$c = -\frac{1}{3}$



b) $f(x) = \frac{x+5}{x-1}$ over the interval $[-3, 5]$

Discontinuity is at $x = 1$ and

$-3 < 1 < 5$

No MVT

Increasing vs. Decreasing

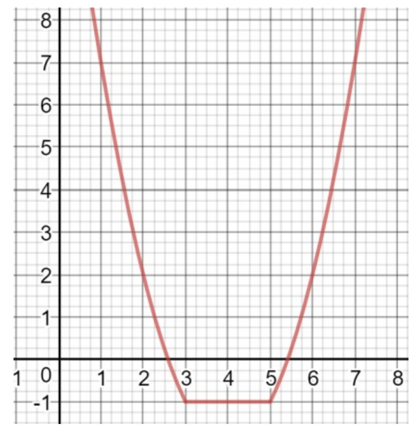
Definitions of Increasing and Decreasing Functions

A function f is increasing on an interval if for any two numbers x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) < f(x_2)$.

A function f is decreasing on an interval if for any two numbers x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) > f(x_2)$.

Example 2

a) Over what interval is the function increasing? decreasing? constant?



Inc $(5, \infty)$

constant $(3, 5)$

Dec $(-\infty, 3)$

b) What is the value of the derivative when the function is increasing? decreasing? constant?

Inc $f'(x) > 0$

constant. $f'(x) = 0$

Dec. $f'(x) < 0$

Test for Increasing and Decreasing Functions

Let f be a function that is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b).

1. If $f'(x) > 0$ for all x in (a, b), then f is increasing on (a, b)
2. If $f'(x) < 0$ for all x in (a, b), then f is decreasing on (a, b)
3. If $f'(x) = 0$ for all x in (a, b), then f is constant on (a, b)

Guidelines for Finding Intervals on which a function is Increasing or Decreasing

Let f be a function that is continuous on the closed interval $[a, b]$. To find the intervals on which f is increasing or decreasing use the following steps:

1. Find the critical points of f in the interval (a, b) and use these numbers to create a sign chart.
2. Use the signs of the derivative to determine whether the function is increasing or decreasing.

Your sign chart is NOT a justification to your response. Your response should be written in words …
"The function is increasing (or decreasing) on the interval (c, d) since $f'(x) > 0$ (or $f'(x) < 0$)."

Example 3 Find the intervals on which $f(x) = 4x^3 - 15x^2 - 18x + 7$ is increasing or decreasing. Also, find all local extrema and justify each response.

$f'(x) = 12x^2 - 30x - 18$

$0 = 6(2x^2 - 5x - 3)$

$0 = 6(2x + 1)(x - 3)$

$x = -1/2 \quad x = 3$

$+ \quad 0 \quad - \quad 0 \quad + \quad f'(x)$

$-1/2 \qquad 3$

Inc. $(-\infty, -\frac{1}{2}), (3, \infty)$

Dec: $(-\frac{1}{2}, 3)$

Rel max at $x = -\frac{1}{2}$
$f'(x)$ changes + to −

Rel min at $x = 3$
$f'(x)$ changes − to +

Example 4 Find the intervals on which $f(x) = (x^2 - 9)^{\frac{2}{3}}$ is increasing or decreasing. Also, find all local extrema and justify each response.

$f'(x) = \frac{2}{3}(x^2-9)^{-\frac{1}{3}}(2x)$

$f'(x) = \frac{4x}{3\sqrt[3]{x^2-9}}$

$f'(x) = 0$ if $x = 0$

$f'(x)$ is undefined $x = 3, -3$

$$\begin{array}{ccccccc} - & & + & & - & & + \\ & -3 & & 0 & & 3 & \end{array} \quad f'(x)$$

Inc $(-3,0), (3,\infty)$ $f'(x) > 0$

Dec $(-\infty,-3), (0,3)$ $f'(x) < 0$

Local (Rel) min at $x = -3$ and $x = 3$

local (Rel) max at $x = 0$

Anti-Derivatives

Example 5 Suppose you were told that $f'(x) = 2x - 1$.

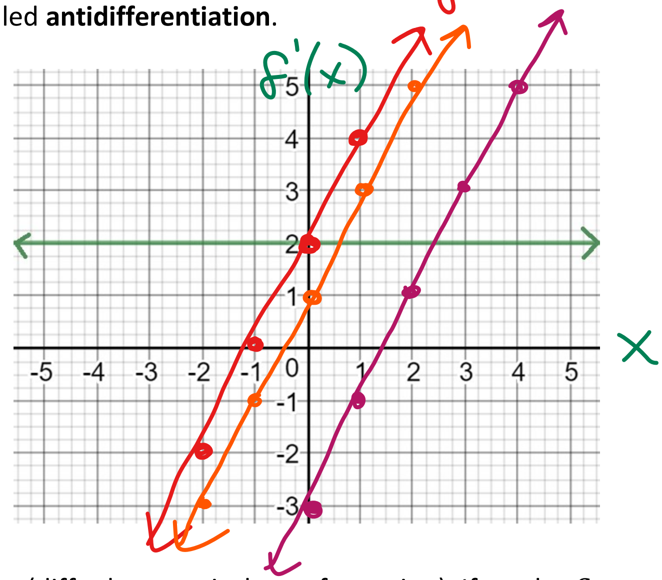
a) What could $f(x)$ be? Is there more than one answer?

$f(x) = x^2 - x + c$

yes (+c could be any constant)

The process of finding the function from the derivative is called **antidifferentiation**.

b) Suppose the graph of $f'(x)$ is given to the right. Draw at least 3 possibilities for $f(x)$. Remember, if $f'(x)$ is given, then the y-values are the slopes of the original function at those x-values.



$f'(x) = 2$

$f(x) = 2x + c$

$2x+2$, $2x+1$, $2x-3$

The three functions you drew should only differ by a constant (differ by a vertical transformation). If you let C represent this constant, then you can represent the family of all antiderivatives of $f'(x)$ to be $f(x) = 2x + C$.

c) If you were told that $f(3) = -2$, what would the value of C be?

$f(x) = 2x + C$

$-2 = 2(3) + c$

$-2 = 6 + c$

$-8 = c$

$y = 2x - 8$

If a function has one antiderivative then it has many antiderivatives that all differ by a constant. Unless you know something about the original function, you cannot determine the exact value of that constant, but the $+C$ must be included in your answer.