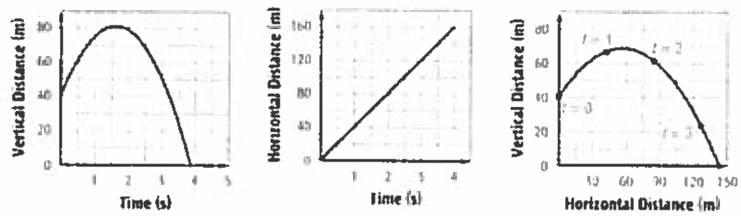
BC Calculus Parametric Equations and Calculus Notesheet Name: Key

Until now, you have been representing a graph by a single equation involving two variables. In this section, we look at situations in which three variables are used to represent a curve in a plane. Consider the graphs below, each of which models different aspects of what happens when a certain object is thrown into the air.



The graph on the left shows the vertical distance the object travels as a function of time. The middle graph shows the object's horizontal distance as a function of time. The graph on the right shows the object's vertical distance as a function of its horizontal distance. Each of these graphs and their equations tell part of what is happening in this situation, but not the whole story. To express the position of the object, both horizontally and vertically, we can use parametric equations. The equations below both represent the graph shown on the right.

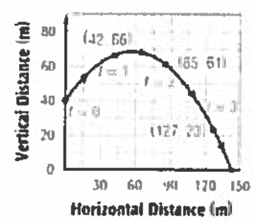
Rectangular Equation

$$y = -\frac{2}{225}x^2 + x + 40$$

Parametric Equations

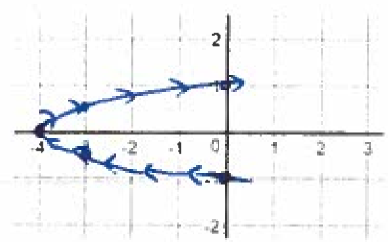
$$x = 30\sqrt{2}t$$
$$y = -16t^2 + 30\sqrt{2}t + 40$$

From the parametric equations, we can now determine where the object was at a given time by evaluating the horizontal and vertical components for t. For example, when $t = 0$, the object was at $(0, 40)$. The variable t is called the parameter.
The graph shown is plotted over the time interval $0 \le t \le 4$. Plotting points in the order of increasing values of t traces the curve in a specific direction called the orientation of the curve. This orientation is indicated by arrows on the curve as shown.



Example 1 Without a calculator, make a table, and sketch the curve, indicating the direction of your graph.
$x = t^2 - 4$ and $y = \frac{t}{2}$, $-2 \le t \le 3$.

t	$x = t^2 - 4$	$y = \frac{t}{2}$	
-2	0	-1	
-1	-3	-1/2	
0	-4	0	
1	-3	1/2	
2	0	1	
3	5	3/2	no room



Finding a rectangular equation that represents the graph of a set of parametric equations is called eliminating the parameter. This basically means that you solve one equation for t and plug in to the other equation, giving you an equation in terms of only x and y.

Example 2 Eliminate the parameter in the following parametric equations. Verify on your calculator.
$x = t^2 - 4$ and $y = \frac{t}{2}$, $-2 \le t \le 3$.

$y = \frac{t}{2}$

$2y = t$

$x = (2y)^2 - 4$

$x = 4y^2 - 4$ or $\boxed{x + 4 = 4y^2}$

What do you notice about the graph of $x = 4t^2 - 4$ and $y = t$, $-1 \le t \le 1.5$?

It's the same graph

What do you notice about the graph of $x = 4(2\sin t + 1)^2 - 4$ and $y = 2\sin t + 1$, $-1.571 \le t \le 0.253$?

It's the same graph

$9 = \frac{1}{t+1} \rightarrow 9t+9=1 \quad t = \frac{-8}{9}$ $\qquad 2 = \frac{1}{\sqrt{t+1}} \rightarrow 4 = \frac{1}{t+1} \rightarrow 4t+4=1 \quad t = -3/4$

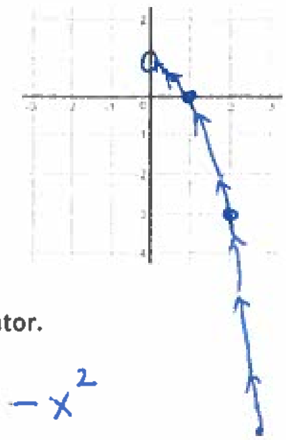
Example 3 Make a table and sketch the curve represented by the following parametric equations.

$x = \frac{1}{\sqrt{t+1}}, y = \frac{t}{t+1}, t > -1.$

t	$x=\frac{1}{\sqrt{t+1}}$	$y=\frac{t}{t+1}$
-1	goes to ∞	goes to $-\infty$
0	1	0
1	$\sqrt{2}/2$	$1/2$
2	$\sqrt{3}/3$	$2/3$
3	$1/2$	$3/4$

too small

t	x	y
$-8/9$	3	-8
$-3/4$	2	-3
3	$1/2$	$3/4$



b) Eliminate the parameter in the parametric equations above. Verify on your calculator.

$x^2 = \frac{1}{t+1}$

$t+1 = \frac{1}{x^2}$

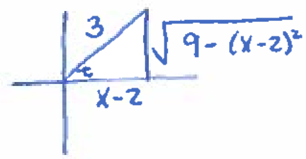
$t = \frac{1}{x^2} - 1 \rightarrow y = \frac{\frac{1}{x^2}-1}{\frac{1}{x^2}-1+1} \rightarrow \frac{\frac{1}{x^2}-1}{\frac{1}{x^2}} = 1 - x^2$

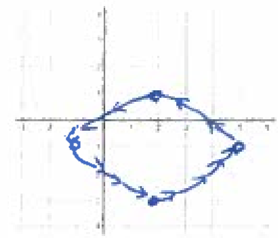
$\boxed{y = 1 - x^2}$

Example 4 Make a table and sketch the curve represented by the following parametric equations.

$x = 2 + 3\cos t, y = -1 + 2\sin t$

t	$x=2+3\cos t$	$y=-1+2\sin t$
0	5	-1
$\pi/2$	2	1
π	-1	-1
$3\pi/2$	2	-3
2π	5	-1





b) Eliminate the parameter in the parametric equations above. Verify on your calculator.

*much easier to solve for $\cos t$ & $\sin t$ and use $\cos^2 t + \sin^2 t = 1$

$x - 2 = 3\cos t$

$\frac{x-2}{3} = \cos t$

$\cos^{-1}\left(\frac{x-2}{3}\right) = t$

$y = -1 + 2\sin\left(\cos^{-1}\left(\frac{x-2}{3}\right)\right)$

$y = -1 + 2\frac{\sqrt{9-(x-2)^2}}{3}$

$(y+1)^2 = \left(\frac{2}{3}\sqrt{9-(x-2)^2}\right)^2$

$(y+1)^2 = \frac{4}{9}(9-(x-2)^2)$

$(y+1)^2 = 4 - \frac{4}{9}(x-2)^2$

$\frac{4}{9}(x-2)^2 + (y+1)^2 = 4$

$\frac{(x-2)^2}{9} + \frac{(y+1)^2}{4} =$

Time to Bring on the Calculus!

Derivatives of Parametric Equations

If a smooth curve C (basically, it does not have any pointy places) is given by the equations $x = f(t)$ and $y = g(t)$, then the slope of C at the point (x, y) is given by

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Where $\frac{dx}{dt} \neq 0$, and the second derivative is given by

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dt}\left[\frac{dy}{dx}\right] \cdot \frac{dt}{dx} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}}$$

To find the first derivative, find the derivative of y with respect to t and divide by the derivative of x with respect to t. To find the second derivative, take the derivative of the first derivative with respect to t and divide by the derivative of x with respect to t.

$\rightarrow 6t^{\frac{3}{2}} - 2t^{\frac{1}{2}}$

Example 5 Given $x = 2\sqrt{t}$, $y = 3t^2 - 2t$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ and evaluate $t = 1$.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{6t-2}{\frac{1}{\sqrt{t}}} \boxed{= \sqrt{t}\,(6t-2)}$$

$$\frac{dy}{dt} = 6t - 2 \qquad \frac{dx}{dt} = \frac{1}{\sqrt{t}}$$

$$\left.\frac{dy}{dx}\right|_{t=1} = (\sqrt{1})(6(1)-2) \quad \boxed{=4}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\frac{dy}{dx}}{\frac{dx}{dt}} = \frac{9t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{t^{-\frac{1}{2}}}$$

$$\boxed{= 9t - 1} \qquad \left.\frac{d^2y}{dx^2}\right|_{t=1} \boxed{=8}$$

Example 6 Given $x = 4\cos t$, $y = 3\sin t$, write an equation of the tangent line to the curve at the point where $t = \frac{3\pi}{4}$.

$$y - y_1 = m(x - x_1) \rightarrow y - y_1 = \frac{dy}{dx}(x - x_1)$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad \frac{dy}{dt} = 3\cos t \qquad \frac{dx}{dt} = -4\sin t$$

$$\rightarrow \frac{3\cos t}{-4\sin t} = -\frac{3}{4}\cos t \Rightarrow -\frac{3}{4}\cos\frac{3\pi}{4} \rightarrow -\frac{3}{4}(-1) = \frac{3}{4}$$

$$x = 4\cos\frac{3\pi}{4} \rightarrow 4\left(-\frac{\sqrt{2}}{2}\right) = -2\sqrt{2} \qquad y = 3\sin\frac{3\pi}{4} \rightarrow 3\left(\frac{\sqrt{2}}{2}\right) \rightarrow \frac{3\sqrt{2}}{2}$$

$$\boxed{y - \frac{3\sqrt{2}}{2} = \frac{3}{4}(x + 2\sqrt{2})}$$

Example 7 Find all points of horizontal and vertical tangency given $x = t^2 + t$ and $y = t^2 - 3t + 5$.

$$\frac{dy}{dx} = 0 \qquad \frac{dy}{dt} = 2t - 3 \qquad \frac{dx}{dt} = 2t + 1$$

$$\frac{2t-3}{2t+1} = 0 \qquad t = \frac{3}{2} \quad \text{H.T}$$

$$x = \left(\frac{3}{2}\right)^2 + \frac{3}{2} = \frac{15}{4} \qquad y = \left(\frac{3}{2}\right)^2 - 3\left(\frac{3}{2}\right) + 5 = \frac{11}{4}$$

@ $\left(\frac{15}{4}, \frac{11}{4}\right)$

$$t = -\frac{1}{2} \quad \text{V.T.} \qquad x = \left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right) = -\frac{1}{4} \qquad y = \left(-\frac{1}{2}\right)^2 - 3\left(-\frac{1}{2}\right) + 5$$

$$\frac{1}{4} + \frac{3}{2} + 5 = 6\tfrac{3}{4}$$

@ $\left(-\frac{1}{4}, 6\frac{3}{4}\right)$

Parametric Arc Length

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Is the length of the arc from $t = a$ to $t = b$.

Example 8 Find the arc length of the given curve if $x = t^2$, $y = 4t^3 - 1$, $0 \le t \le 1$.

$$\frac{dx}{dt} = 2t \qquad \frac{dy}{dt} = 12t^2$$

$$L = \int_0^1 \sqrt{(2t)^2 + (12t^2)^2}\, dt$$

$$\boxed{\approx 4.149}$$

Particle Motion with Parametric Equations

$x'(t) = \dfrac{dx}{dt}$	the rate at which the x-coordinate is changing with respect to t or the velocity of an object in the horizontal direction.
$y'(t) = \dfrac{dy}{dt}$	the rate at which the y-coordinate is changing with respect to t or the velocity of an object in the vertical direction.
$(x(t), y(t))$	the position at any time t.
$(x'(t), y'(t))$	the velocity at any time t.
$(x''(t), y''(t))$	the acceleration at any time t.
$\dfrac{dy}{dx}$	the rate of change of y with respect to x or the slope of the tangent line to the curve.
$\dfrac{d^2y}{dx^2}$	the rate of change of the slope of the curve with respect to x.
$\sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}$	The speed of a particle
$\int_a^b \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}\, dt$	the distance traveled by a particle from $t = a$ to $t = b$.

Example 9 A particle moves in the xy-plane so that any time t, $t \geq 0$, the position of the particle is given by $x(t) = t^3 + 4t^2$, $y(t) = t^4 - t^3$.

a) Find the parametric equations representing the velocity of the particle. Then find the horizontal and vertical components of the velocity at $t = 1$.

$x'(t) = 3t^2 + 8t$ $\qquad x'(1) = 3 + 8 = 11$

$y'(t) = 4t^3 - 3t^2$ $\qquad y'(1) = 4 - 3 = 1$

$\boxed{v(1) = (11, 1)}$

b) Find the speed of the particle at $t = 1$.

speed $= \sqrt{(x'(t))^2 + (y'(t))^2} = \sqrt{11^2 + 1^2} = \boxed{\sqrt{122}}$

c) Find the parametric equations representing the acceleration of the particle. Then find the horizontal and vertical components of the acceleration at $t = 1$.

$x''(t) = 6t + 8$ $\qquad x''(1) = 14$

$y''(t) = 12t^2 - 6t$ $\qquad y''(1) = 6$

$\boxed{a(1) = (14, 6)}$

d) Find the distance the particle travels from time $t = 2$ to $t = 5$.

distance traveled $= \int_2^5 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$ $\boxed{\approx 536.462}$