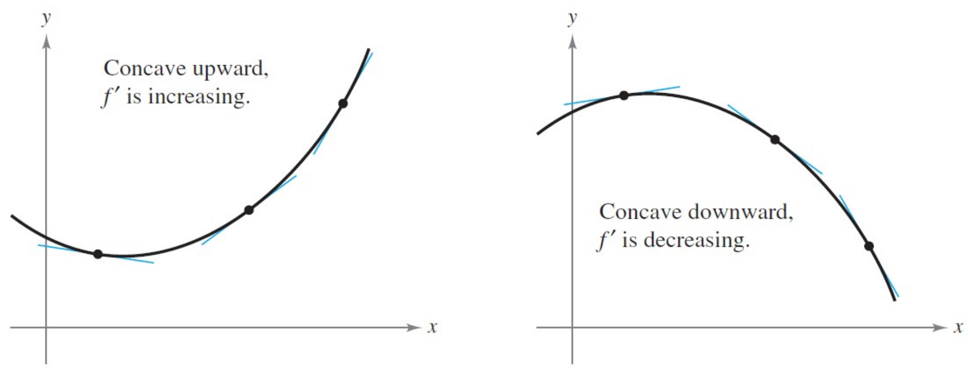
AB Calculus: Concavity and the Second Derivative Test   Name: ____________________

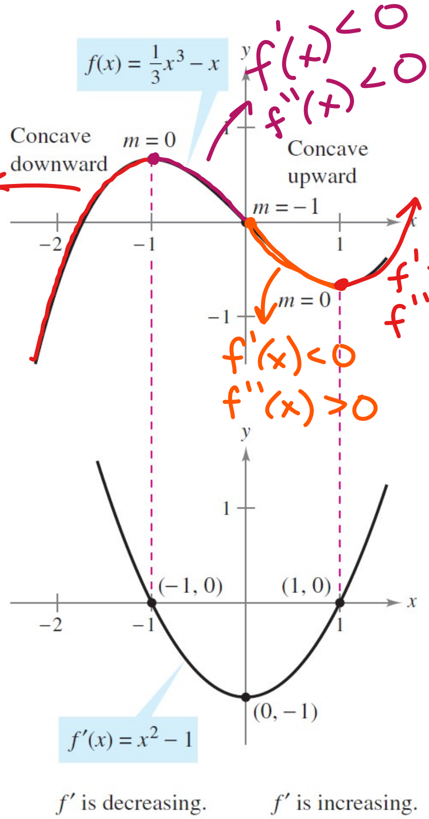
You have already seen that locating the intervals in which a function is increasing and decreasing helps to describe its graph. In this section, you will see how locating the intervals in which the derivative is increasing or decreasing can be used to determine when a function is curving upward or curving downward.

Definition: Concavity

Let f be differentiable on an open interval I. The graph of f is concave upward on I if f' is increasing on the interval and concave downward on I if f' is decreasing on the interval.



(a) The graph of f lies above its tangent lines. **(b)** The graph of f lies below its tangent lines.



To find the open intervals on which the graph of a function f is concave upward or concave downward, you need to find the intervals on which f' is increasing or decreasing. For instance, the graph of

$$\boxed{f(x) = \frac{1}{3}x^3 - x} \quad f'(x) = x^2 - 1 \quad x^2 - 1 = 0$$

$$(x+1)(x-1) = 0 \quad x = -1, 1$$

is concave downward over the open interval $(-\infty, 0)$ because $f'(x) = x^2 - 1$ is decreasing there. Similarly, the graph of f is concave upward on the interval $(0, \infty)$ because f' is increasing over $(0, \infty)$.

$+ \;\; -1 \;\; - \;\; 1 \;\; + \quad f'(x) \qquad f''(x) \;\; - \;\; 0 \;\; +$

The following theorem shows how to use the second derivative of a function f to determine intervals on which the graph of f is concave upward or concave downward. $f''(x) = 2x \quad 2x = 0 \quad x = 0$

Test for Concavity
Let f be a function whose second derivative exists on an open interval I.
1. If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.
2. If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

- If the second derivative is positive, then the first derivative is __increasing__ and the original function is __concave up__.
- If the second derivative is negative, then the first derivative is __decreasing__ and the original function is __concave down__.

Guidelines for Determining Concavity

1. Find the second derivative of the function.
2. Locate the x-values where $f''(x) = 0$ and $f''(x)$ is undefined.
3. Make a sign chart for the second derivative using the x-values you found in part 2.
4. In areas where the second derivative is positive, $f(x)$ is concave up. In areas where $f''(x)$ is negative, $f(x)$ is concave down.

Example 1 Let f be the function defined below. Determine the open intervals on which the graph of f is concave upward, then find the intervals on which f is concave downward.

$$f(x) = 4x^3 + 21x^2 + 36x - 20$$

$f'(x) = 12x^2 + 42x + 36$

$f''(x) = 24x + 42$

$0 = 24x + 42$

$24x = -42$

$x = -\frac{42}{24}$

$x = \frac{-7}{4}$

f'': $-$ on the left of $-7/4$, $+$ on the right of $-7/4$

f is concave down $(-\infty, -7/4)$

f is concave up $(-7/4, \infty)$

Which makes the point of inflection occur at $x = -7/4$

Example 2 Let f be the function defined below. Determine the open intervals on which the graph of f is concave upward, then find the intervals on which f is concave downward.

$$f(x) = \frac{x^2 + 1}{x^2 - 4}$$

$f'(x) = \frac{(x^2-4)(2x) - (x^2+1)(2x)}{(x^2-4)^2}$

$\Rightarrow 2x^3 - 8x - 2x^3 - 2x$

$\Rightarrow \frac{-10x}{(x^2-4)^2}$

$f''(x) = \frac{(x^2-4)^{\cancel{2}}(-10) - (-10x)\,2\cancel{(x^2-4)}^{1}(2x)}{(x^2-4)^{\cancel{4}\,3}}$

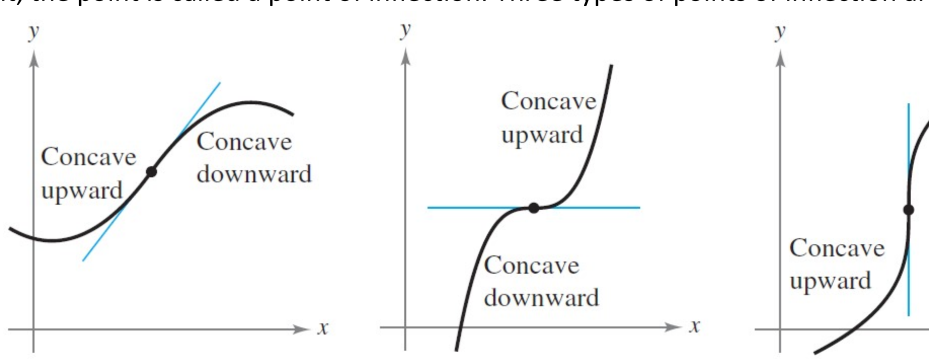
$= \frac{-10(x^2-4-4x^2)}{(x^2-4)^3}$

$f''(x) = \frac{-10(-3x^2-4)}{(x^2-4)^3}$

$-3x^2 - 4 = 0$
$-3x^2 = 4$ → No solutions here → Look at bottom
$x^2 - 4 = 0$
$x = 2, -2$

Points of Inflection

The graph in example 2 has two points at which the concavity changes. If the tangent line to the graph exists at such a point, the point is called a point of inflection. Three types of points of inflection are shown below.



f'': $+$ | -2 | $-$ | 2 | $+$

$f(x)$ is concave up $(-\infty, -2)$ $(2, \infty)$

$f(x)$ is concave down $(-2, 2)$

Definition: Point of Inflection (POI)

Let f be a function that is continuous on an open interval and let c be a point in the interval. If the graph of f has a tangent line at this point $(c, f(c))$, then this point is a point of inflection of the graph of f if the concavity of f changes from upward to downward or downward to upward at the point.

If $(c, f(c))$ is a point of inflection of the graph of f, then either $f''(c) = 0$ or $f''(c)$ is undefined.

Example 3: Determine the points of inflection of the graph of $f(x) = x^4 - 4x^3$

$f'(x) = 4x^3 - 12x^2$

$f''(x) = 12x^2 - 24x$

$0 = 12x^2 - 24x$

$0 = 12x(x-2)$

$x = 0, 2$

f'': $+$ | 0 | $-$ | 2 | $+$

Point of inflection at $x = 0$ and $x = 2$ since $f''(x)$ changes sign

The Second Derivative Test

In addition to testing for concavity, the second derivative can be used to perform a simple test for relative maxima and minima. The test is based on the fact that if the graph of a function f is concave upward on an open interval containing c, and $f'(c) = 0$, $f(c)$ must be a relative minimum. Similarly, if the graph of a function f is concave downward on an open interval containing c, and $f'(c) = 0$, $f(c)$ must be a relative maximum of f.

The Second Derivative Test

Let f be a function such that $f'(c) = 0$ and the second derivative of f exists on an open interval containing c.

1. If $f''(c) > 0$, then f has a relative minimum at $(c, f(c))$.
2. If $f''(c) < 0$, then f has a relative maximum at $(c, f(c))$

If $f''(c) = 0$, the test fails. That is, f may have a relative maximum, a relative minimum, or neither. In such cases, you can use the first derivative test.

Example 4: Find the relative extrema for the function $f(x) = -3x^5 + 5x^3$ using the second derivative test.

$f'(x) = -15x^4 + 15x^2 \longrightarrow f''(x) = -60x^3 + 30x$

$0 = -15x^2(x^2 - 1)$

$0 = -30x(2x^2 - 1)$

$f'(x) = 0$ at $x = 0, 1, -1$

$f''(0) = 0 \rightarrow$ no idea

$f''(-1) = +$ → I have a rel. min.

$f''(1) = -$ → I have a rel max

Note: Usually, you can choose whether you use the first derivative test to find relative extrema or the second derivative test. However, there are ways to ask a question so that you have to use the second derivative test. For example, being given that $f(x)$ is continuous at $x = 2$, $f'(2) = 0$, and $f''(2) = -3$ is enough information to determine that $f(x)$ has a relative max at $x = 2$ because of the second derivative test. You would be unable to make that determination if you only knew the first derivative test.