Sequences and Series Notes

1a. *[2 marks]*

In an arithmetic sequence, the first term is 8 and the second term is 5.

Find the common difference.

$5 - 8 = -3$

1b. *[2 marks]*

Find the tenth term.

$a_{10} = 8 + (-3)(10-1)$

$8 - 3(9)$

$= -19$

$u_n = u_1 + d(n-1)$

1c. *[2 marks]*

Find the sum of the first ten terms.

$S_n = \frac{n}{2}(a_1 + a_n)$

$S_{10} = \frac{10}{2}(8 + (-19))$

$5(-11) = -55$

2a. *[2 marks]*

In an arithmetic sequence, the first term is 3 and the second term is 7.

Find the common difference.

$7 - 3 = 4$

2b. *[2 marks]*

Find the tenth term.

$u_n = u_1 + d(n-1)$

$u_{10} = 3 + 4(10-1) = 39$

2c. *[2 marks]*

Find the sum of the first ten terms of the sequence.

$S_{10} = \frac{10}{2}(3+39) = 5(42) = 210$

3. *[6 marks]*

Three consecutive terms of a geometric sequence are $x-3$, 6 and $x+2$.

Find the possible values of x.

$r = \frac{6}{x-3}$

$r = \frac{x+2}{6}$

$x = 7, -6$

$\frac{6}{x-3} = \frac{x+2}{6}$

$36 = (x-3)(x+2)$

$36 = x^2 - 3x + 2x - 6$

$-36 \qquad\qquad -36$

$0 = x^2 - x - 42$

$0 = (x-7)(x+6)$

4. *[6 marks]*

An arithmetic sequence has the first term $\ln a$ and a common difference $\ln 3$.

The 13th term in the sequence is $8 \ln 9$. Find the value of a.

$a_{13} = a_1 + d(13-1)$

$8 \ln 9 = \ln a + 12 \ln 3$

$\ln 9^8 = \ln a + \ln 3^{12}$

$\ln (3^2)^8 = \ln (a \cdot 3^{12})$

$\frac{3^{16}}{3^{12}} = \frac{a\, 3^{12}}{3^{12}}$

$\boxed{a = 3^4 = 81}$

$\ln(ab) = \ln a + \ln b$

$\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

$\ln(a^b) = b \ln a$

$\log_a b = x \;/\; a^x = b$

5a. *[4 marks]*

??

The sums of the terms of a sequence follow the pattern

→ integer

$S_1 = 1 + k,\ S_2 = 5 + 3k,\ S_3 = 12 + 7k,\ S_4 = 22 + 15k,\ \ldots,$ where $k \in \mathbb{Z}$.

Given that $\boxed{u_1 = 1 + k}$, find u_2, u_3 and u_4.

$S_2 = u_1 + u_2$

$5 + 3k = 1 + k + u_2$

$-1 - k = -1 - k$

$\boxed{4 + 2k = u_2}$

$S_3 = S_2 + u_3$

$12 + 7k = 5 + 3k + u_3$

$-5 - 3k = -5 - 3k$

$\boxed{7 + 4k = u_3}$

$S_4 = S_3 + u_4$

$22 + 15k = 12 + 7k + u_4$

$-12 - 7k = -12 - 7k$

$\boxed{10 + 8k = u_4}$

5b. *[4 marks]*

Find a general expression for u_n.

$$\boxed{u_n = 1 + 3(n-1) + 2^{n-1} \cdot k}$$

6a. *[1 mark]*

The first three terms of an infinite geometric sequence are 32, 16 and 8.

Write down the value of r. $\frac{16}{32} = \frac{1}{2}$

6b. *[2 marks]*

Find u_6.

$u_n = u_1 \cdot r^{n-1}$

$u_6 = 32 \left(\frac{1}{2}\right)^5$

$= \frac{32}{32} = 1$

6c. *[2 marks]*

Find the sum to infinity of this sequence. $|r| < 1$

$$S_\infty = \frac{a_1}{1-r} = \frac{32}{1-1/2} = \frac{32}{1/2} = 64$$

7a. *[2 marks]*

The first three terms of a infinite geometric sequence are $m-1,\ 6,\ m+4$, where $m \in \mathbb{Z}$.

Write down an expression for the common ratio, r.

$\frac{6}{m-1}$ or $\frac{m+4}{6}$

7b. *[2 marks]*

Hence, show that m satisfies the equation $m^2+3m-40=0$.

$\frac{6}{m-1} = \frac{m+4}{6}$

$36 = (m-1)(m+4)$

$36 = m^2 + 4m - m - 4$

$0 = m^2 + 3m - 40$

7c. *[3 marks]*

Find the two possible values of m.

$(m+8)(m-5) = 0$

$m = -8, 5$

7d. *[3 marks]*

Find the possible values of r.

$\frac{6}{5-1} = \frac{6}{4} = \frac{3}{2}$

$\frac{6}{-8-1} = \frac{6}{-9} = \frac{-2}{3}$

7e. *[3 marks]*

The sequence has a finite sum.

$|r| < 1$

State which value of r leads to this sum **and** justify your answer.

$r = -\frac{2}{3}$

7f. *[3 marks]*

The sequence has a finite sum.

Calculate the sum of the sequence.

$S_\infty = \frac{a_1}{1-r} = \frac{-9}{1-(-\frac{2}{3})} = \frac{-9}{1+\frac{2}{3}} = \frac{-9}{\frac{3+2}{3}} = \frac{-27}{5}$

8a. *[1 mark]*

Consider the infinite geometric sequence $3,\ 3(0.9),\ 3(0.9)^2,\ 3(0.9)^3,\ \ldots.$

Write down the 10th term of the sequence. Do not simplify your answer.

$3(9)^9$

8b. *[4 marks]*

Consider the infinite geometric sequence $3,\ 3(0.9),\ 3(0.9)^2,\ 3(0.9)^3,\ \ldots.$

Find the sum of the infinite sequence.

$S_\infty = \frac{3}{1-9} = \frac{3}{1} = 30$

9a. *[2 marks]*

The first two terms of an infinite geometric sequence are $u_1 = 18$ and $u_2 = 12\sin^2\theta$, where $0 < \theta < 2\pi$, and $\theta \neq \pi$.

Find an expression for r in terms of θ.

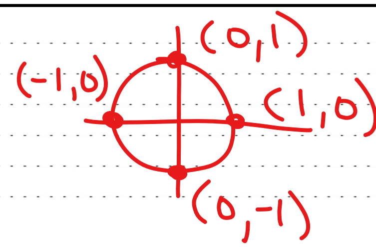
$r = \frac{12\sin^2\theta}{18} = \frac{2\sin^2\theta}{3}$

9b. *[3 marks]*

Find the possible values of r.

$0 < r \leq \frac{2}{3}$

$0 < \sin^2\theta \leq 1$



9c. *[4 marks]*

Show that the sum of the infinite sequence is $\frac{54}{2+\cos(2\theta)}$.

$$\frac{a_1}{1-r} = \frac{18}{1-\frac{2}{3}\sin^2\theta}$$

$$\frac{54}{3-2\sin^2\theta} \rightarrow \frac{54}{2+1-2\sin^2\theta}$$

$$\rightarrow \frac{54}{2+\cos(2\theta)}$$

9d. *[6 marks]*

Find the values of θ which give the greatest value of the sum.

$\frac{a_1}{1-r}$ greatest value

if $r = \frac{2}{3}$

$$\frac{2}{3} = \frac{2}{3}\sin^2\theta$$

$$1 = \sin^2\theta$$

$\sin\theta = 1$ $\quad$ $\sin\theta = -1$

$\theta = \frac{\pi}{2}$ $\quad$ $\theta = \frac{3\pi}{2}$

10a. *[2 marks]*

An arithmetic sequence has $u_1 = \log_c(p)$ and $u_2 = \log_c(pq)$, where $c > 1$ and $p,\ q > 0$.

Show that $d = \log_c(q)$.

$d = \log_c(pq) - \log_c(p)$

$= \log_c\left(\frac{pq}{p}\right) = \log_c(q)$

10b. *[6 marks]*

Let $p = c^2$ and $q = c^3$. Find the value of $\sum_{n=1}^{20} u_n$. sigma $\Rightarrow$ sum

$u_1 + u_2 + \quad u_{20}$

$p = c^2$

$\log_c(p) = 2$

$u_1 = 2$

$u_1 = \log_c(p)$

$c^{u_1} = p$

$u_1 = 2$

$q = c^3$

$u_2 = \log_c(pq)$

$c^{u_2} = pq$

$c^{u_2} = c^2 c^3$

$c^{u_2} = c^5$

$u_2 = 5$

$d = u_2 - u_1 = 5 - 2 = 3$

$S_{20} = \frac{20}{2}(2 + u_{20})$

$= 10(2 + 59)$

$= 20 + 590$

$S_{20} = 610$

$u_{20} = 2 + 3(20 - 1)$

$2 + 3(19)$

$= 59$

11a. *[3 marks]*

The first three terms of a geometric sequence are $\ln x^{16}, \ln x^8, \ln x^4$, for $x > 0$.

Find the common ratio.

$r = \frac{\ln x^8}{\ln x^{16}} = \frac{8 \ln x}{16 \ln x} = \frac{1}{2}$

11b. *[5 marks]*

$S_\infty \rightarrow \frac{a_1}{1-r}$

Solve $\sum_{k=1}^{\infty} 2^{5-k} \ln x = 64$.

$\frac{a_1}{1-r} = 64$

$\frac{16 \ln x}{1 - 1/2} = 64$

$\frac{16 \ln x}{1/2} = 64$

$32 \ln x = 64$

$\ln x = 2$

$x = e^2$

12. *[6 marks]*

The sum of an infinite geometric sequence is 33.25. The second term of the sequence is 7.98. Find the possible values of r.