AB Calculus: Rates of Change and Tangent Lines **Name:** ______________________________

The World Record Basketball Shot

A group called How Ridiculous became YouTube famous when they successfully made a basket from the top of Tasmania's Gordon Dam, 415 feet above the ground, breaking the World Record for the highest basketball shot on June 14, 2015. This record stood until a group called Dude Perfect made a basket from the top of a building at Texas Christian University (TCU) from a height of 533 feet, a World Record that still stands today. Use the information from the Dude Perfect World Record Shot to answer the following questions.

Part 1

a) If an object is dropped from an initial height, h_0, we can use the position function $s(t) = -16t^2 + h_0$ to model the height, s, in feet of an object that has fallen for t seconds. What is the position function for the basketball as the shot makes its way to the ground?

$$S(t) = -16t^2 + 533$$

b) A moving body's average velocity during an interval of time is found by dividing the total change in position by the elapsed amount time. If the moving body's position is modeled by $s(t)$ and the interval of time is $[a, b]$, the average velocity can be found using the formula: $\frac{s(b)-s(a)}{b-a}$. Find the basketball's average velocity for the first 3 seconds of flight. Does this formula remind you of anything? → SLOPE

$$\frac{s(3)-s(0)}{3-0} = \frac{389-533}{3} = -48 \text{ ft/s}$$

c) Find the basketball's average velocity between t = 2 and t = 3 seconds.

$$\frac{s(3)-s(2)}{3-2} = \frac{389-469}{1} = -80 \text{ ft/s}$$

d) Find basketball's velocity at the instant t = 3 seconds.

$$\frac{S(3)-S(3)}{3-3} = \frac{0}{0} \text{ Indeter}$$

e) What complication(s), if any, did you encounter when answering any of the previous questions?

The calculation @ exactly t=3 is indeterminate

Part 2

a) Find the average velocity between t = 2.5 and t = 3 seconds.

$$\frac{389-433}{3-2.5} = -88 \text{ ft/s}$$

b) Find the average velocity between t = 2.9 and t = 3 seconds.

$$\frac{389-398.44}{3-2.9} = -94.4 \text{ ft/s}$$

c) Find the average velocity between t = 2.99 and t = 3 seconds.

$$\frac{389-389.9584}{3-2.99} = -95.84 \text{ ft/s}$$

d) Find the average velocity between t = 2.999 and t = 3 seconds.

$$\frac{389-389.095984}{3-2.999} = -95.984 \text{ ft/s}$$

e) What is the basketball's velocity approaching at t = 3 seconds?

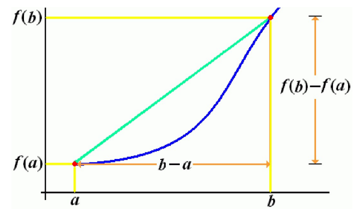
$$-96 \text{ ft/s}$$

Secant Lines and Average Rate of Change

A **secant** line is a line joining two points on a function. Finding the slope of the secant line through the points $P_1(a, f(a))$ and $P_2(b, f(b))$ will tell you the average rate of change over the interval $[a, b]$. This was the process you were doing in the previous basketball shot scenario when you found the slope during intervals of time. The process to find the average rate of change is no different than the process you have used to find the slope of a line since Math 1, we will just write it a little differently.

Average Rate of Change

$$\frac{f(b) - f(a)}{b - a}$$



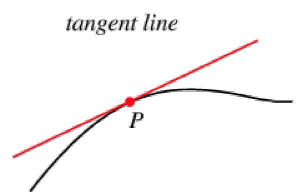
Example 1: Find the average rate of change of $f(x) = x^3 - x$ over the interval $[1, 3]$.

A ROC over $[1,3]$ $= \dfrac{f(3)-f(1)}{3-1}$ $\quad f(3) = 3^3 - 3 = 24$ $\quad f(1) = 1^3 - 1 = 0$ $\quad \dfrac{24-0}{3-1} = 12$

Tangent Lines and Instantaneous Rate of Change

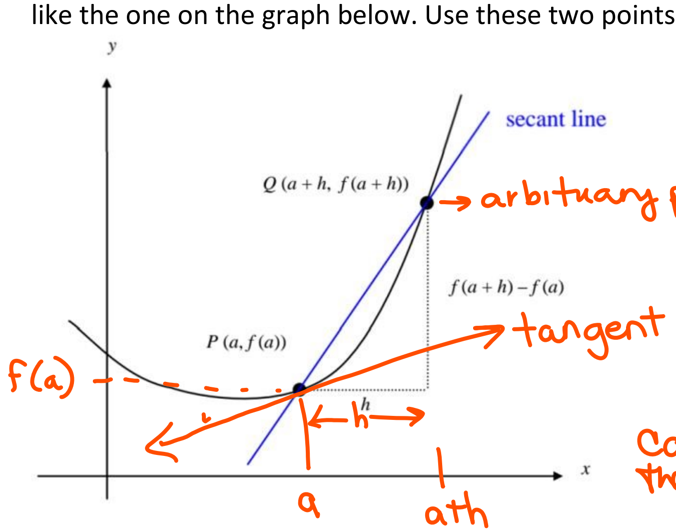
A **tangent** line is a line that touches a function at only one point. Finding the slope of the tangent line at a point will tell you the instantaneous rate of change at that point. Since we only have one point, using the average rate of change formula would cause a problem. If we try using the single point $P(a, f(a))$ in our average rate of change formula, we will get indeterminate form.

$$\frac{f(a) - f(a)}{a - a} = \frac{0}{0}$$



We can approximate the slope of the tangent line using a secant line. Just like in the basketball shot scenario, the closer the second point is to the point of tangency, the better the estimation for the slope of the tangent line at the point.

However, if we want to prove the slope of the curve is the value found in our approximation, we are going to have to find it algebraically. What if we add a very small amount (we will call this amount h), to our point $P(a, f(a))$ to create an arbitrary second point $Q(a + h, f(a + h))$? Then we would have a secant line scenario like the one on the graph below. Use these two points in the average rate of change formula to find the result.



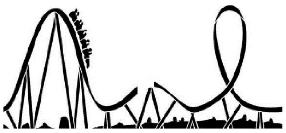
arbituary point (then pull it closer to the point of interest)

by closing the gap between the two independent values

Calculate the average rate of change $= \dfrac{f(a+h) - f(a)}{a+h - a} = \dfrac{f(a+h) - f(a)}{h}$

If we substitute 0 for h, then we get indeterminate form again. However, if we find the limit as h approaches 0, then we can use algebraic techniques to evaluate the limit and find the slope of the tangent line at that single point, or the instantaneous rate of change!

Slope of a Curve at a Point



The slope of a line is always constant. The slope of a curve is constantly changing. Think of a curve as a roller coaster that you are riding. If for some reason the track were to just disappear, you would go flying off in the direction that you were traveling at that last instant before the track disappeared. The direction that you flew off to would be the slope of the curve at that point. To find this slope, we can find the limit as h approaches 0 of the result you got when finding the slope of the tangent line.

Slope of a Curve at a Point

The slope of the curve $y = f(x)$ at the point $P(a, f(a))$ is the number

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Provided the limit exists.

- The expression $\frac{f(a+h)-f(a)}{h}$ is called the **difference quotient**. the horizontal displacement
- Your goal will be to simplify the difference quotient, then evaluate the limit as h approaches 0.
- The difference quotient is simplified when you have divided out the h in the denominator in a correct way.

Example 2: Before we use this definition, be sure to be comfortable with the notation $f(a+h)$.

a) If $f(x) = \frac{1}{x}$, what is $f(a)$? $f(a+h)$?

$f(a) = \frac{1}{a}$ $\quad f(a+h) = \frac{1}{a+h}$

b) If $f(x) = x^2 - 4x$, what is $f(a)$? $f(a+h)$?

$f(a) = a^2 - 4a$ $\quad f(a+h) = (a+h)^2 - 4(a+h) = a^2 + 2ah + h^2 - 4a - 4h$

c) If $f(x) = \sqrt{4x+1}$, what is $f(a)$? $f(a+h)$?

$f(a) = \sqrt{4a+1}$ $\quad f(a+h) = \sqrt{4(a+h)+1}$

Example 3: Find the slope of the curve $f(x) = x^2$ at the point $(2, 4)$.

$m = \lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$f(x+h) = (x+h)^2 = x^2 + 2xh + h^2$

$\lim\limits_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$

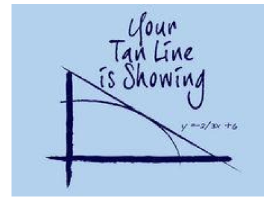
$\lim\limits_{h \to 0} \frac{2xh + h^2}{h}$ $\quad \frac{\cancel{h}(2x+h)}{\cancel{h}}$

$\lim\limits_{h \to 0} 2x + h = 2x$

The slope of $f(x) = x^2$ is $2x$

Slope of $f(x)$ at $(2,4)$ is $2(2) = 4$

Finding the Equation of a Tangent Line to a Curve



If the slope of a curve at a point is the slope of the tangent line through the point, then we can find the equation of the line as well. Remember, to write the equation of a line, we need two things: a point on the line and the slope. You will generally be given the point, or at least one coordinate of it. We can find the slope by taking the limit as h approaches 0 of the difference quotient.

Example 4: Find the equation of the tangent line to the curve $f(x) = x^2$ at the point $(2, 4)$. x_1 y_1

point-slope

$y - y_1 = m(x - x_1)$

$x_1 = 2$

$y_1 = 4$

$m = 2x$ and $x_1 = 2$ so $m = 2(2) = 4$

$\boxed{\text{Tangent Line} \quad y - 4 = 4(x-2)}$

Recall that there are limits that fail to exist. Since the slope of a tangent line is a limit, the slope might not exist because the limit doesn't exist (discontinuity, vertical asymptote, oscillating function) or because the tangent line has a vertical slope.

For a tangent line that exists, but has an undefined slope, this definition does not quite fit. To cover the possibility of a vertical tangent line, we can use the following definition.

If f is continuous at a and $$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \pm\infty$$ The vertical line $x = a$ is a vertical tangent line to the graph of f.

Example 5: What types of graphs would have vertical tangent lines?

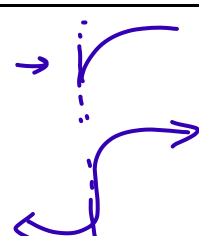
graphs with vertical asymptotes

circles

ellipses

$\sqrt{x} \rightarrow$

$\sqrt[3]{x}$



Example 6: Back to the basketball shot example. Let $f(x) = -16x^2 + 533$. $\boxed{\text{Find the slope of the curve at } x = 3.}$

$$m = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$f(x+h) = -16(x+h)^2 + 533$$
$$= -16(x^2 + 2xh + h^2) + 533$$
$$= -16x^2 - 32xh - 16h^2 + 533$$

$$\lim_{h\to 0} \frac{-16x^2 - 32xh - 16h^2 + 533 - (-16x^2 + 533)}{h}$$

$-16x^2 - -16x^2$ cancels

$533 - 533$ cancels

$$\lim_{h\to 0} \frac{-32xh - 16h^2}{h}$$

Now cancel "h"

$$\lim_{h\to 0} -32x - 16h = -32x$$

$\boxed{-32x \text{ is the slope of } f(x) \text{ at any } x\text{-value}}$

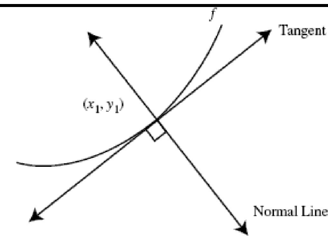
This would be the instantaneous velocity of the basketball at exactly 3 seconds!

at $x = 3$, $-32x = -32(3) = -96$

Normal Lines

The **normal line** to a curve at a point is the line perpendicular to the tangent at that point.

→ perpendicular   opposite, reciprocal slope



To find the slope of the normal line, take the opposite reciprocal of the slope of the tangent line.

Example 7: Let $f(x) = \frac{1}{x+1}$.

a) Find the slope of the curve at $x = a$. (I'll use just "x")

$$m = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \qquad f(x+h) = \frac{1}{x+h+1}$$

$$\lim_{h\to 0} \frac{\frac{(x+1)}{(x+1)}\frac{1}{(x+h+1)} - \frac{1}{(x+1)}\frac{(x+h+1)}{(x+h+1)}}{h}$$

$$\lim_{h\to 0} \frac{x+1-x-h-1}{h(x+1)(x+h+1)} \to \lim_{h\to 0} \frac{-h}{h(x+1)(x+h+1)} \to \lim_{h\to 0} \frac{-1}{(x+1)(x+h+1)}$$

$$= \frac{-1}{(x+1)^2}$$

b) Find the slope of the curve at $x = 2$.

Plug in $x=2$ $\qquad \frac{-1}{(2+1)^2} = -\frac{1}{9}$

b) Write the equation of the tangent line to the curve at $x = 2$.

Tangent line required a point (x, y) and slope
↳ $f(x)$

Point $(2, \frac{1}{3})$
m $-\frac{1}{9}$

$$y - \frac{1}{3} = -\frac{1}{9}(x-2)$$

c) Write the equation of the normal line to the curve at $x = 2$.

$$y - \frac{1}{3} = 9(x-2)$$