BC Calculus Taylor Series Notesheet B Name: ______________________________

There are 3 Maclaurin series that show up so often that it is a good idea to commit them to memory.

Maclaurin Series for $f(x) = e^x$

$$f(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Maclaurin Series for $f(x) = \sin x$

$$f(x) = \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Maclaurin Series for $f(x) = \cos x$

$$f(x) = \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

If these series have been committed to memory, then they can be conveniently manipulated to find series representations of similar functions.

You can manipulate these series by using the following techniques:

Techniques for Manipulating Series

1. Substituting into a series for x
2. Multiply or divide the series by a constant and/or variable.
3. Add or subtract 2 series.
4. Differentiate or integrate a series, which may change the interval, but not the radius of convergence.
5. Recognize the series as the sum of a geometric power series.

$\sin(x) \to$ g.t $\to \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$c=0$

Example 1 Find a Maclaurin series for $f(x) = \sin(x^2)$. Find the first four nonzero terms and the general term.

$$x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!}$$

G.T for $\sin(x^2) \to \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!}$

Example 2 Find a Maclaurin series for $f(x) = x\cos(x)$. Find the first four nonzero terms and the general term.

$$x - \frac{x^3}{2!} + \frac{x^5}{4!} - \frac{x^7}{6!}$$

G.T for $\cos(x) \to \frac{(-1)^n x^{2n}}{(2n)!}$

G.T. for $x \cdot \cos x \to \frac{(-1)^n x \, x^{2n}}{(2n)!} \to \frac{(-1)^n x^{2n+1}}{(2n)!}$

Example 3 Find a Maclaurin series for $f(x) = 4e^{x^2}$ Find the first four nonzero terms and the general term.

old $e^x \to 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \quad + \frac{x^n}{n!}$

$$4e^{x^2} \to 4 + 4x^2 + \frac{4x^4}{2!} + \frac{4x^6}{3!} + \cdots + \frac{4x^{2n}}{n!}$$

Example 4 Find the first 3 nonzero terms in the Maclaurin series for $f(x) = e^x \sin x$.

$e^x \to 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$ $\sin x \to x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$

$$x - \frac{x^3}{3!} + \cancel{\frac{x^6}{5!}} + x^2 + \frac{x^3}{2!} \to -\frac{x^3}{3!} + \frac{3x^3}{3!} = \frac{2x^3}{3!} = \frac{x^3}{3}$$

Final ans

$$x + x^2 + \frac{x^3}{3}$$

Series Free Response Examples

$c = 0$

Example 5 The function g has derivatives of all orders, and the Maclaurin series for g is

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+3} = \frac{x}{3} - \frac{x^3}{5} + \frac{x^5}{7} - \cdots.$$

a) Using the ratio test, determine the interval of convergence of the Maclaurin series for g.

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{\cancel{2(n+1)+1}\ 2n+3}}{\cancel{2(n+1)+3}\ \ 2n+5} \cdot \frac{2n+3}{(-1)^n x^{2n+1}} \right|$$

Logic (A) $\lim_{n\to\infty} \left| \frac{2n+3}{2n+5} \right| = 1$

(B) $\frac{x^{2n+3}}{x^{2n+1}} = x^2$

(C) $\left| \frac{(-1)^{n+1}}{(-1)^n} \right| = 1$

$r = 1$

$|x^2| < 1 \quad -1 \le x \le 1$

Check $x = 1$ into $g \to \frac{(-1)^n}{2n+3}$ ✓

Check $x = -1 \to \frac{(-1)^n (-1)^{2n+1}}{(-1)^{3n+1}} \to$ stays alt

b) The Maclaurin series for g evaluated at $x = \frac{1}{2}$ is an alternating series whose terms decrease in absolute value to 0. The approximation for $g\left(\frac{1}{2}\right)$ using the first two nonzero terms of this series is $\frac{17}{120}$. Show that this approximation differs from $g\left(\frac{1}{2}\right)$ by less than $\frac{1}{200}$.

Plug in $x = \frac{1}{2}$ to the 3rd term $\to \frac{x^5}{7}$

$\left(\frac{1}{2}\right)^5 \frac{1}{7} \to \frac{1}{32} \cdot \frac{1}{7} \to \frac{1}{224} < \frac{1}{200}$

$\left| g\left(\frac{1}{2}\right) - \frac{17}{120} \right| < \frac{1}{224} < \frac{1}{200}$

c) Write the first three nonzero terms and the general term of the Maclaurin series for $g'(x)$.

$\frac{1}{3} - \frac{3x^2}{5} + \frac{5x^4}{7} \qquad \frac{(-1)^n (2n+1)\, x^{2n}}{2n+3}$

Example 6 A function f has derivatives of all orders at $x = 0$. Let $P_n(x)$ denote the nth-degree Taylor polynomial for f about $x = 0$.

↳ Maclaurin Series

a) It is known that $f(0) = -4$ and $P_1\left(\frac{1}{2}\right) = -3$. Show that $f'(0) = 2$.

$P_1(x) = -4 + f'(0)\, x$

$-3 = -4 + f'(0)\, \frac{1}{2}$ add 4

$1 = f'(0)\, \frac{1}{2}$

multiply by 2 $\to 2 = f'(0)$

b) It is known that $f''(0) = -\frac{2}{3}$ and $f'''(0) = \frac{1}{3}$. Find $P_3(x)$.

$-4 + 2x - \frac{2}{3} \cdot \frac{x^2}{2!} + \frac{1}{3} \cdot \frac{x^3}{3!} \to P_3(x)$ is an approx. for $f(x)$

c) The function h has first derivative given by $h'(x) = f(2x)$. It is known that $h(0) = 7$. Find the third-degree Taylor polynomial for h about $x = 0$.

initial condition

→ Integrate from center to x

1st find the 1st few terms of $f(2x) \to$ replace x w/ $2x$

$\int_0^x h'(x) \to \int_0^x -4 + 4x - \frac{2}{3} \cdot \frac{4x^2}{2!} + \frac{1}{3} \cdot \frac{8x^3}{3!} \longrightarrow \int_0^x -4 + \int_0^x 4x - \int \frac{4}{3} x^2 + \frac{4x^3}{9}$

$$\boxed{7 - 4(x-0) + \frac{4x^2}{2} - \frac{4x^3}{9}}$$