AP Calculus Vector Calculus Day 2 Homework

Name: Key

1. Find dy/dx and d^2y/dx^2 in terms of t for the parametric function $x = cos^3t$, $y = sin^3t$ when $t = \frac{\pi}{4}$.

$(\cos t)^3 \quad (\sin t)^3$

(1) $\frac{dx}{dt} = -3\cos^2 t\sin t \rightarrow \frac{dx}{dt}\Big|_{t=\pi/4} = -3\left(\frac{\sqrt{2}}{2}\right)^2\left(\frac{\sqrt{2}}{2}\right) = \frac{-3\sqrt{2}}{4}$

$\frac{dy}{dt} = 3\sin^2 t\cos t \rightarrow \frac{dy}{dt}\Big|_{t=\frac{\pi}{4}} = 3\left(\frac{\sqrt{2}}{2}\right)^2\left(\frac{\sqrt{2}}{2}\right) = \frac{3\sqrt{2}}{4}$

so $\frac{dy}{dx}\Big|_{t=\frac{\pi}{4}} = -1$

(2) $\frac{dy}{dx} = \frac{3\sin^2 t\cos t}{-3\cos^2 t\sin t} = -\frac{\sin t}{\cos t} = -\tan t \quad \frac{d}{dt}-\tan t = -\sec^2 t \quad \frac{-\sec^2 t}{-3\cos^2 t\sin t}\Big|_{t=\frac{\pi}{4}} = \frac{-2}{\frac{-3\sqrt{2}}{4}} =$

$\frac{d^2y}{dx^2}\Big|_{t=\frac{\pi}{4}} = \frac{8}{3\sqrt{2}}$

2. Find the points where the tangent to the curve $x = 3\cos(t)$, $y = 3\sin(t)$ is horizontal and vertical.

$\frac{dx}{dt} = -3\sin t \qquad 3\cos t = 0 \qquad t = \frac{\pi}{2}, \frac{3\pi}{2}$

H.T: $(3(0), 3(1)) \rightarrow (0,3)$

$(3(0), 3(-1)) \rightarrow (0,-3)$

$\frac{dy}{dt} = 3\cos t \qquad -3\sin t = 0 \qquad t = 0, \pi$

V.T: $(3\cdot 1, 3\cdot 0)$

$(3,0)$ or $(-3,0)$

3. Find the length of the curve $x = e^{-t}\cos(t)$, $y = e^{-t}\sin(t)$ $\quad 0 \le t \le \frac{\pi}{2}$.

* Remember you don't have to find derivative by hand

$L = \int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \approx 1.120$

carefully put everything in your calculator!

4. Find the unit vector in the direction of u and v and the angle between u and v if u = <4, 3> and v = <12, -5>.

$4\cdot 12 + 3\cdot -5$

$48 - 15$

$33 = (5)(13)\cos\theta$

$\theta = 59.490°$

$|u| = 5$

$|v| = 13$

unit vector in direction of u $\left\langle \frac{4}{5}, \frac{3}{5} \right\rangle$

u.v. in dir. of v $\left\langle \frac{12}{13}, \frac{-5}{13} \right\rangle$

5. A curve in the plane is defined parametrically by the equations $x = t^3 + t$ and $y = t^4 + 2t^2$. An equation of the line tangent to the curve at $t = 1$ is:

$x = 2 \quad y = 3$

A) $y = 2x$

B) $y = 8x$

C) $y = 2x - 1$

D) $y = 4x - 5$

E) $y = 8x + 13$

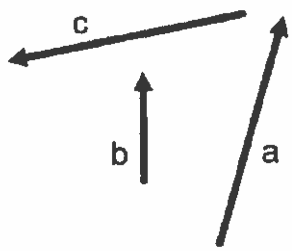
$\frac{dx}{dt} = 3t^2 + 1 \qquad \frac{dy}{dt} = 4t^3 + 4t \qquad \frac{4(1)^3 + 4(1)}{3(1)^2 + 1} = \frac{8}{4} = 2$

$y - 3 = 2(x - 2)$

6. Find the component form of the vector of length 5 that makes an angle $\theta = 135°$ with the positive x-axis.

$$\langle 5\cos 135°, 5\sin 135° \rangle = \langle -\frac{5\sqrt{2}}{2}, \frac{5\sqrt{2}}{2} \rangle$$

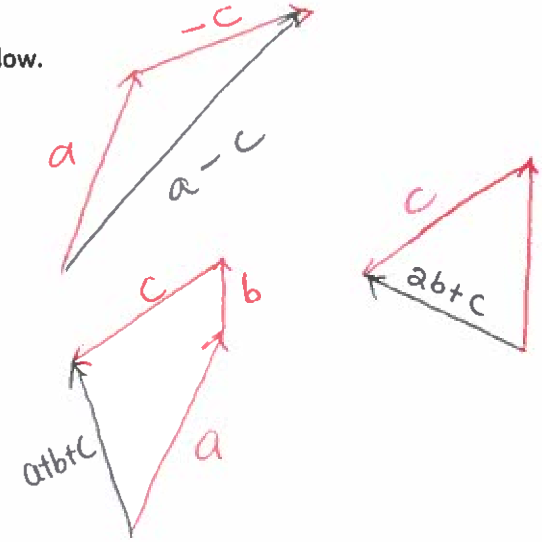
7. Sketch the indicated vector using the vectors below.



a) a - c

b) 2b + c

c) a + b + c



8. Find the unit vectors that are tangent and normal to the curve $x = t$, $y = \sqrt{25 - t^2}$ where t = 3.

$$\frac{dx}{dt} = 1$$

$$\frac{dy}{dt} = \frac{1}{2}(25-t^2)^{-\frac{1}{2}}(-2t)$$

$$\left.\frac{dy}{dt}\right|_{t=3} = \frac{1}{2}(16)^{-\frac{1}{2}}(-6)$$

$$= \frac{-3}{4}$$

vector $= \langle 1, -\frac{3}{4} \rangle$

$$\sqrt{1^2 + \left(\frac{-3}{4}\right)^2}$$

$$= \sqrt{\frac{16}{16} + \frac{9}{16}}$$

$$= \frac{5}{4}$$

un.vector $\langle \frac{1}{5/4}, \frac{-3/4}{5/4} \rangle$

T: $\langle \frac{4}{5}, \frac{-3}{5} \rangle, \langle \frac{-4}{5}, \frac{3}{5} \rangle$

N: $\langle \frac{3}{5}, \frac{4}{5} \rangle, \langle \frac{-3}{5}, \frac{-4}{5} \rangle$

9. An airplane is headed 32° North of West. Its speed in still air is 900 km/h. The wind at the plane's altitude is from blowing in the north east direction at 100 km/h. What is the true direction of the plane? What is the speed of the plane with respect to the ground?

$\to$ 180 - 32 = 148° from origin

Airplane = $\langle 900\cos 148°, 900\sin 148° \rangle$

Wind = $\langle 100\cos 45°, 100\sin 45° \rangle$

True $\approx \langle -692.533, 547.638 \rangle$

$\theta = \tan^{-1}\left(\frac{547.638}{-692.533}\right)$ Put in Q2 :)

$\downarrow$

38.336° N of W

Speed $\approx$ 882.898 km/hr

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1. For each of the following, r(t) is the position vector of a particle. Find the velocity and acceleration vectors, as well as the speed and direction of motion at the given value of t.

   a. $r(t) = (2\cos t)i + (3\sin t)j, \quad t = \frac{\pi}{2}$

$v(t) = r'(t) = (-2\sin t)i + (3\cos t)j$

$a(t) = r''(t) = (-2\cos t)i + (-3\sin t)j$

$r'(\frac{\pi}{2}) = -2i + 0j$

speed = 2

direction is West

   b. $r(t) = 2\ln(t+1)i + (t^2)j, \quad t = 1$

$v(t) = r'(t) = \left(\frac{2}{t+1}\right)i + (2t)j$

$a(t) = r''(t) = \left(\frac{-2}{(t+1)^2}\right)i + 2j$

$r'(1) = 1i + 2j$

speed $= \sqrt{5}$ direction $63.435°$ N of E

$\tan^{-1}(\frac{2}{1})$

2. Find an equation for the line that is (a) tangent and (b) normal to the curve r(t) at the point determined by the given value of t.

$r(t) = \sin(t)i + (t^2 - \cos(t))j, \quad t = 0$

$r(0) = 0i + (-1)j$

$r'(t) = (\cos t)i + (2t + \sin t)j$

$r'(0) = 1i + 0j$

$\frac{dy}{dx} = \frac{0}{1} = 0$ $y + 1 = 0(x - 0)$

or $y = -1$

3. Evaluate each integral.

   a) $\int_1^2 \left[(6-6t)i + \left(3\sqrt{t}\right)j\right] dt$

$\left[6t - 3t^2\right]_1^2 i + \left[2t^{\frac{3}{2}}\right]_1^2 j$

$\left(6(2) - 3(2)^2\right) - \left(6(1) - 3(1)^2\right)$

$0 - 3 = -3$

$2(2)^{\frac{3}{2}} - 2(1)^{\frac{3}{2}} = 2\sqrt{8} - 2$

$\boxed{-3i + (2\sqrt{8} - 2)j}$

   b) $\int [(\sec(t)\tan(t))i + (\tan(t))j]\, dt$ $\rightarrow \int \frac{\sin}{\cos} \rightarrow$ $u = \cos$, $du = -\sin$ $\rightarrow -\int \frac{1}{u} \rightarrow -\ln|u|$

$(\sec t)i + (-\ln|\cos t|)j + C$

4. Solve the initial value problem for r as a vector function of t.

   a. $\frac{dr}{dt} = \frac{3}{2}(t+1)^{\frac{1}{2}}i + e^{-t}j, \quad r(0) = 0$

$r = \left(\frac{3}{2}\left(\frac{2}{3}\right)(t+1)^{\frac{3}{2}}\right)i + \left(-e^{-t}\right)j + C$

$r = (t+1)^{\frac{3}{2}} i + (-e^{-t})j + C \rightarrow 0i + 0j = (0+1)^{\frac{3}{2}} i + (-e^{-0})j + C$

$C = -1i + 1j \rightarrow r = ((t+1)^{\frac{3}{2}} - 1)i + (-e^{-t} + 1)j$

b. $\frac{d^2r}{dt^2} = -32j$, $r(0) = 100i$, $r'(0) = 8i + 8j$

$\frac{dr}{dt} = (-32t)j + C$

$(-32\cdot 0)j + C = 8i + 8j$

$\frac{dr}{dt} = 8i + (-32t + 8)j$

$r = (8t)i + (-16t^2 + 8t)j + C$

$100i = 0i + (0)j + C$

$r(t) = (8t + 100)i + (-16t^2 + 8t)j$

5. The position of a particle moving in the xy-plane is given by the parametric equations $x = t^3 - 3t^2$ and $y = 2t^3 - 3t^2 - 12t$. For what values of t is the particle at rest?

no vertical or horizontal motion

A) -1 B) 0 C) 2 D) $-1, 2$ E) $-1, 0, 2$

$t = 0, 2 \rightarrow 2$ for both

$t = -1, 2$

$\frac{dx}{dt} = 3t^2 - 6t$ $\frac{dy}{dt} = 6t^2 - 6t - 12$

$3t^2 - 6t = 0$ $3t(t-2) = 0$

$6t^2 - 6t - 12 = 0$ $6(t+1)(t-2) = 0$

$6(t^2 - t - 2) = 0$

6. A particle moves on a plane curve so that at any time $t > 0$ its x-coordinate is $t^3 - t$ and its y-coordinate is $(2t-1)^3$. The acceleration vector of the particle at $t = 1$ is

A) $(0, 1)$ B) $(2, 3)$ C) $(2, 6)$ D) $(6, 12)$ E) $(6, 24)$

$v(t) = \langle 3t^2 - 1, 3(2t-1)^2(2) \rangle$

$\rightarrow 6(2t-1)^2$

$a(t) = \langle 6t, 12(2t-1)(2) \rangle$

$a(1) = \langle 6, 24 \rangle$

7. (1984 BC2, p. 538 #32) The path of a particle is given for time $t > 0$ by the parametric equations $x = t + \frac{2}{t}$ and $y = 3t^2$.

$t + 2t^{-1}$

(a) Find the coordinates of each point on the path where the velocity of the particle in the x direction is zero.

$\frac{dx}{dt} = 1 - \frac{2}{t^2}$

$1 - \frac{2}{t^2} = 0$ $-\frac{2}{t^2} = -1$

$-t^2 = -2$

$t^2 = 2$

$t = \pm\sqrt{2}$ (just $\sqrt{2}$ b/c domain)

$x = \sqrt{2} + \frac{2}{\sqrt{2}}$ $y = 3(\sqrt{2})^2$

$\left(\sqrt{2} + \frac{2}{\sqrt{2}}, 6\right)$

(b) Find $\frac{dy}{dx}$ when $t = 1$.

$\frac{dy}{dt} = 6t$

$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{6t}{1 - \frac{2}{t^2}}$

$\left.\frac{dy}{dx}\right|_{t=1} = \frac{6}{1 - \frac{2}{1}} = \frac{6}{1-2} = -6$

(c) Find $\frac{d^2y}{dx^2}$ when $y = 12$.

$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\frac{dy}{dx}}{\frac{dx}{dt}} = \frac{\frac{(t^2-2)(18t^2) - (6t^3)(2t)}{(t^2-2)^2}}{1 - \frac{2}{t^2}}$

1st $\frac{dy}{dx} = \frac{6t}{\frac{t^2-2}{t^2}} = \frac{6t^3}{t^2-2}$ for ease

$y = 3t^2 = 12$

$t^2 = 4$ or $t = 2$

$\frac{(2)(72) - (48)(4)}{4} \div \frac{1}{2} = -24$