2 A particle P is at the origin O at time $t = 0$. The particle moves with constant velocity and arrives at the point Q with position vector $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 20 \\ -8 \end{pmatrix}$ m 4 seconds later. Find

a the velocity of P

b the position of P if it continues moving past this point with the same velocity for 6 more seconds.

Another particle T is moving with constant velocity $(12\mathbf{i} - 5\mathbf{j})\,\text{m s}^{-1}$. It passes through the point A whose position vector is $(4\mathbf{i} - \mathbf{j})$ m at $t = 0$.

c Find the speed of the particle.

d Find the distance of T from O when $t = 3$ s.

e Will the two particles collide?

3 In this question distances are given in kilometres and time in hours. A unit vector represents a displacement of 1 km. At 3 p.m. a man is standing on the top of a cliff looking out to sea and observing two ships traveling. Ship A's position relative to a point on the shore is given by $3\mathbf{i} + 3\mathbf{j}$ and it is traveling with a velocity of $4\mathbf{i} + 3\mathbf{j}$. Ship B's position is given by $4\mathbf{i} + 3\mathbf{j}$ and it is traveling with a velocity of $3\mathbf{i} + 3\mathbf{j}$. Find

a the time at which the two ships will collide if one does not change course

b the point at which the ships will collide.

EXAM-STYLE QUESTION

4 The position of two helicopters X and Y at time t seconds are given by the formulae

$$\mathbf{r}_x = \begin{pmatrix} 11 \\ 3 \\ -3 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} \text{ and } \mathbf{r}_y = \begin{pmatrix} 1 \\ -7 \\ -2 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ 9 \end{pmatrix} \text{ respectively.}$$

Distances are given in metres.

a Find the speed of the two helicopters.

b Show that the two helicopters do not meet.

c Find the distance between the helicopters when $t = 10$.

No calc 1–8

Review exercise

1 Prove using a vector method that the points $A(1, 2, 3)$, $B(-2, 3, 5)$ and $C(7, 0, -1)$ are collinear.

EXAM-STYLE QUESTION

2 Show the points A, B and C with position vectors $5\mathbf{i} - \mathbf{j} + 6\mathbf{k}$, $2\mathbf{i} + 2\mathbf{j}$ and $-3\mathbf{i} - 5\mathbf{j} + 8\mathbf{k}$ respectively form a right-angled triangle.

EXAM-STYLE QUESTION

3 Given that $\mathbf{a} = \begin{pmatrix} 5 \\ -1 \\ -3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix}$, show that the vectors $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$ are perpendicular.

4 Two lines with equations $\mathbf{r}_1 = \begin{pmatrix} 0 \\ 6 \\ -1 \end{pmatrix} + s\begin{pmatrix} 7 \\ 3 \\ 1 \end{pmatrix}$ and $\mathbf{r}_2 = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$ intersect at the point P. Find the coordinates of P.

EXAM-STYLE QUESTIONS

5 A triangle has its vertices at $A(-2, 4)$, $B(1, 7)$ and $C(-3, 2)$.

a Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

b Find $\overrightarrow{AB} \cdot \overrightarrow{AC}$.

c Show that $\cos B\hat{A}C = \frac{3}{\sqrt{2}\sqrt{5}}$.

6 Two lines L_1 and L_2 are given by $\mathbf{r}_1 = \begin{pmatrix} 6 \\ 2 \\ -3 \end{pmatrix} + s\begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$ and $\mathbf{r}_2 = \begin{pmatrix} 0 \\ -12 \\ 7 \end{pmatrix} + t\begin{pmatrix} -1 \\ 11 \\ -3 \end{pmatrix}$.

a P is the point on L_1 when $s = 4$. Find the position vector of P.

b Show that P is also on L_2.

7 The line L_1 has vector equation $\mathbf{r} = \begin{pmatrix} 2 \\ -3 \\ -3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$.

L_2 is parallel to L_1 and passes through the point B(2, 2, 4).

a Write down a vector equation for L_2 in the form $\mathbf{r} = \mathbf{a} + s\mathbf{b}$.

A third line L_3 is perpendicular to L_1 and is represented by $\mathbf{r} = \begin{pmatrix} 3 \\ 11 \\ 7 \end{pmatrix} + q\begin{pmatrix} 7 \\ x \\ 1 \end{pmatrix}$.

b Show that $x = -3$.

c Find the coordinates of the point C, the intersection of L_1 and L_3.

d Find $\overrightarrow{BC}$.

e Find $|\overrightarrow{BC}|$ in the form $a\sqrt{b}$ where a and b are integers to be found.

EXAM-STYLE QUESTION

8 (In this question distances are measured in km and time in hours.) At noon a lighthouse keeper observes two ships A and B.

Ship A's position at time t is given by $\mathbf{r}_1 = \begin{pmatrix} -4 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 17 \end{pmatrix}$.

Ship B's position at time t is given by $\mathbf{r}_2 = \begin{pmatrix} 4 \\ 9 \end{pmatrix} + \mu \begin{pmatrix} -12 \\ 5 \end{pmatrix}$.

a Show that A and B will collide, and find the time when this will occur and the position vector of the point of collision.

In order to prevent collision, at 12:15 ship A changes its direction to $\begin{pmatrix} 16 \\ 17 \end{pmatrix}$.

b Find the distance between A and B at 12:30.

Calc OK 1-6

Review exercise

1 Find the size of the angle between the two vectors $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$. Give your answer to the nearest degree.

EXAM-STYLE QUESTIONS

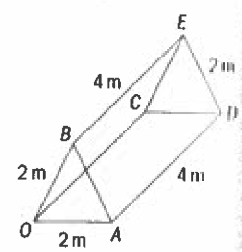
2 The vertices of the triangle PQR are defined by the position vectors

$\overrightarrow{OP} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$, $\overrightarrow{OQ} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$ and $\overrightarrow{OR} = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$. Find

a $\overrightarrow{QR}$ and $\overrightarrow{QP}$ **b** $P\hat{Q}R$ **c** the area of triangle PQR.

3 A tent $OABCDE$ is a triangular prism with a constant cross-section that is an equilateral triangle with sides of 2 m. The tent is 4 m long. The base $OADC$ is horizontal. Support poles are to be laid along the diagonals BC and BD.

Take O as the origin and unit vectors **i** and **j** in the directions of OA and OC respectively, **k** is a unit vector vertically upwards.



a Express these vectors in terms of the unit vectors **i**, **j** and **k**.

 i $\overrightarrow{OC}$ **ii** $\overrightarrow{OB}$ **iii** $\overrightarrow{OD}$

b Hence find vectors $\overrightarrow{BC}$ and $\overrightarrow{BD}$.

c Calculate the values of

 i $|\overrightarrow{BC}|$ **ii** $|\overrightarrow{BD}|$

 iii the scalar product of $\overrightarrow{BC}$ and $\overrightarrow{BD}$.

d Hence find the angle between the support poles.

4 Given that $\mathbf{a} = x\mathbf{i} + (x-2)\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = x^2\mathbf{i} - 2x\mathbf{j} - 12x\mathbf{k}$ where x is a scalar variable, find

a the values of x for which **a** and **b** are perpendicular

b the angle between **a** and **b** when $x = -1$.

5 The points P and Q have position vectors $\begin{pmatrix}1\\-1\\3\end{pmatrix}$ and $\begin{pmatrix}1\\5\\5\end{pmatrix}$ respectively, with respect to an origin O.

a Show that $\overrightarrow{OP}$ is perpendicular to $\overrightarrow{PQ}$.

b Write down the vector equation of the line L_1, which passes through P and Q.

The line L_2 has equation $\boldsymbol{r} = \begin{pmatrix}2\\-1\\2\end{pmatrix} + \mu\begin{pmatrix}1\\-3\\-2\end{pmatrix}$.

c Show that the lines L_1 and L_2 intersect and find the position vector of their point of intersection.

d Calculate, to the nearest degree, the acute angle between the lines L_1 and L_2.

6 All distances in this question are in metres and time is in seconds.

An insect is flying at a constant height. At time $t = 0$, the insect is at point A with coordinates (0, 0, 6). Two seconds later the insect is at point B with coordinates (6, –2, 6).

a Find vector $\overrightarrow{AB}$.

The insect continues to fly in the same direction at the same speed.

b Show that the position vector of the insect at time t is given by

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\0\\6\end{pmatrix} + t\begin{pmatrix}3\\-1\\0\end{pmatrix}.$$

At time $t = 0$, a bird takes off from the ground. The position vector of the bird at time t is given by $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}36\\18\\0\end{pmatrix} + t\begin{pmatrix}-3\\-4\\1\end{pmatrix}$.

c Write down the coordinates of the starting position of the bird.

d Find the speed of the bird.

The bird reaches the insect at point C.

e Find the time the bird takes to reach the insect.

f Find the coordinates of C.

CHAPTER 12 SUMMARY

Vector: basic concepts

- A **vector** is a quantity that has **size** (magnitude) and **direction**. Examples of vectors are displacement and velocity.
- A **scalar** is a quantity that has size but no direction. Examples of scalars are distance and speed.
- The unit vector in the direction of the x-axis is $\mathbf{i}$.

  In two dimensions $\mathbf{i}=\begin{pmatrix}1\\0\end{pmatrix}$ and in three dimensions $\mathbf{i}=\begin{pmatrix}1\\0\\0\end{pmatrix}$
- The unit vector in the direction of the y-axis is $\mathbf{j}$.

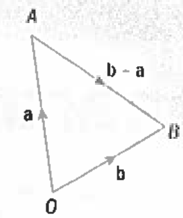
  In two dimensions $\mathbf{j}=\begin{pmatrix}0\\1\end{pmatrix}$ and in three dimensions $\mathbf{j}=\begin{pmatrix}0\\1\\0\end{pmatrix}$
- In three dimensions the unit vector in the direction of the z-axis is $\mathbf{k}$, where

  $\mathbf{k}=\begin{pmatrix}0\\0\\1\end{pmatrix}$
- The vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are called **base vectors**.
- If $\overrightarrow{AB}=\begin{pmatrix}a\\b\end{pmatrix}=a\mathbf{i}+b\mathbf{j}$ then $|\overrightarrow{AB}|=\sqrt{a^2+b^2}$.

  If $\overrightarrow{AB}=\begin{pmatrix}a\\b\\c\end{pmatrix}=a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$ then $|\overrightarrow{AB}|=\sqrt{a^2+b^2+c^2}$.
- Two vectors are **equal** if they have the same direction and the same magnitude; their $\mathbf{i}, \mathbf{j}, \mathbf{k}$ components are equal too, and so their column vectors are equal.
- You can write $\overrightarrow{AB}$ as $-\overrightarrow{BA}$.
- Two vectors are **parallel** if one is a scalar multiple of the other. So, $\overrightarrow{AB}$ and $\overrightarrow{RS}$ are parallel if $\overrightarrow{AB}=k\overrightarrow{RS}$ where k is a scalar quantity. This can also be written as $\mathbf{a}=k\mathbf{b}$.
- The point with coordinates (x, y) has **position vector** $\overrightarrow{OP}=\begin{pmatrix}x\\y\end{pmatrix}=x\mathbf{i}+y\mathbf{j}$.
- To find the **resultant vector** $\overrightarrow{AB}$ between two points A and B, subtract the position vector of A from the position vector of B.

Continued on next page

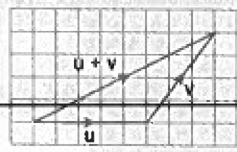
- If $A = (x_1, y_1, z_1)$ then $\mathbf{a} = \overrightarrow{OA} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$
  and if $B = (x_2, y_2, z_2)$ then $\mathbf{b} = \overrightarrow{OB} = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$
  $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$
  $= \overrightarrow{OB} - \overrightarrow{OA}$
  $= \mathbf{b} - \mathbf{a}$
  $= (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}$
  Distance $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$



- A vector of length 1 in the direction of **a** is found by using the formula $\frac{\mathbf{a}}{|\mathbf{a}|}$.
- A vector of length k in the direction of **a** is found by using the formula $k\frac{\mathbf{a}}{|\mathbf{a}|}$.

Addition and subtraction of vectors

- The resultant vector, $\mathbf{u} + \mathbf{v}$, is the third side of the triangle formed when **u** and **v** are placed next to each other head to tail.



- Vectors are subtracted by adding a negative vector.

Scalar product

- **Scalar product**
  If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$ then $\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2$.
  Similarly if $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then
  $\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$.
- The **scalar product** $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta$ where θ is the angle between the vectors.
- For **perpendicular** vectors $\mathbf{a}\cdot\mathbf{b} = 0$.
- For **parallel** vectors $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|$.
- For **coincident** vectors $\mathbf{a}\cdot\mathbf{a} = a^2$.

Vector equation of a line

- The **vector equation** of a line is $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ where **r** is the general position vector of a point on the line, **a** is a given position vector of a point on the line and **b** is a **direction vector** parallel to the line. t is called the parameter.

page

c E.g. $\mathbf{r} = \begin{pmatrix}3\\5\\2\end{pmatrix} + t\begin{pmatrix}-1\\-9\\3\end{pmatrix}$

d E.g. $\mathbf{r} = \begin{pmatrix}0\\0\\1\end{pmatrix} + t\begin{pmatrix}1\\-1\\-1\end{pmatrix}$

3 a E.g. $\mathbf{r} = \begin{pmatrix}-1\\6\end{pmatrix} + t\begin{pmatrix}2\\-3\end{pmatrix}$

b E.g. $\mathbf{r} = \begin{pmatrix}-1\\0\end{pmatrix} + t\begin{pmatrix}2\\5\end{pmatrix}$

c E.g. $\mathbf{r} = \begin{pmatrix}4\\2\\1\end{pmatrix} + t\begin{pmatrix}1\\0\\3\end{pmatrix}$

d E.g. $r = 5\mathbf{k} + t(4\mathbf{i} - \mathbf{k})$

4 a Yes b No

c Yes d No

5 $\mathbf{r} = \begin{pmatrix}2\\4\\5\end{pmatrix} + t\begin{pmatrix}-2\\3\\8\end{pmatrix}$

$p = -2,\ q = 21$

6 E.g. $\mathbf{r} = \begin{pmatrix}-6\\5\end{pmatrix} + t\begin{pmatrix}0\\1\end{pmatrix}$

7 a Coincident

b Perpendicular

c Parallel

d None

e None

8 a 53.6° b 115.2°

10 a i $2\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}$

ii $-2\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}$

b i $|\overrightarrow{OF}| = \sqrt{38}$

ii $|\overrightarrow{AG}| = \sqrt{38}$

iii $\overrightarrow{OF} \cdot \overrightarrow{AG} = 30$

c 37.9°

11 a $\overrightarrow{AB} = 7\mathbf{i} - 8\mathbf{j} + 8\mathbf{k}$

b $\cos O\hat{A}B = \dfrac{-49}{\sqrt{30}\sqrt{117}}$

d $\mu = 3$

e $(22, -19, 22)$

Exercise 12K

1 $(4, 2)$

2 $\begin{pmatrix}\frac{48}{5}\\ -\frac{3}{5}\end{pmatrix}$

3 $\left(\frac{23}{3}, \frac{1}{3}, \frac{2}{3}\right)$

4 $\left(-1, \frac{5}{3}\right)$

6 a $\begin{pmatrix}5\\-8\\15\end{pmatrix}$

b Dot product = 0

7 a $a = 5;\ b = 8$

b $(4, 5, 7)$

c $3\sqrt{10}$

8 a E.g. $\mathbf{r} = \begin{pmatrix}2\\-1\\2\end{pmatrix} + t\begin{pmatrix}1\\-1\\-3\end{pmatrix}$

b $(3, -2, -1)$

c $\sqrt{11}$

d 120.2°

Exercise 12L

1 a $\begin{pmatrix}15\\10\end{pmatrix}$ or 10 km north and 15 km east

b $5\sqrt{13}$ km

2 a $\sqrt{29}\ \mathrm{m\,s^{-1}}$

b $\begin{pmatrix}50\\-20\end{pmatrix}$

c $13\ \mathrm{m\,s^{-1}}$

d $8\sqrt{29}$ m

e They will collide.

3 a 4 p.m. b $7\mathbf{i} + 6\mathbf{j}$

4 a $3\sqrt{2}\ \mathrm{m\,s^{-1}}$ and $\sqrt{86}\ \mathrm{m\,s^{-1}}$

c 51.2 m

non-calc.

Review exercise non-GDC

1 a $\overrightarrow{AB} = \begin{pmatrix}-3\\1\\2\end{pmatrix}$, $\overrightarrow{BC} = \begin{pmatrix}9\\-3\\-6\end{pmatrix}$,

$\overrightarrow{AC} = \begin{pmatrix}6\\-2\\-4\end{pmatrix}$

3 $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = 0$

4 $(7, 9, 0)$

5 a $\overrightarrow{AB} = \begin{pmatrix}3\\3\end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix}-1\\-2\end{pmatrix}$

b $\overrightarrow{AB} \cdot \overrightarrow{AC} = -9$

6 a $\begin{pmatrix}-2\\10\\1\end{pmatrix}$

b $t = 2$

7 a $r = \begin{pmatrix}2\\2\\4\end{pmatrix} + s\begin{pmatrix}1\\3\\2\end{pmatrix}$

b $(4, 8, 8)$

d $\begin{pmatrix}2\\6\\4\end{pmatrix}$

e $2\sqrt{14}$

8 a 12.30 p.m.; $\begin{pmatrix}-2\\11.5\end{pmatrix}$

b 3 km

calc. ok

Review exercise GDC

1 122°

2 a $\overrightarrow{QR} = \begin{pmatrix}-1\\0\\5\end{pmatrix}$, $\overrightarrow{QP} = \begin{pmatrix}0\\-1\\1\end{pmatrix}$

b 46.1°

c 2.60

3 a i $4\mathbf{j}$ ii $\mathbf{i} + \sqrt{3}\mathbf{k}$

iii $2\mathbf{i} + 4\mathbf{j}$

b $\overrightarrow{BC} = -\mathbf{i} + 4\mathbf{j} - \sqrt{3}k$

$\overrightarrow{BD} = \mathbf{i} + 4\mathbf{j} - \sqrt{3}k$

c i $\sqrt{20}$ ii $\sqrt{20}$

iii 18

d 25.8°

4 a $0, 4, -2$

b 82.9°

5 a $\overrightarrow{OP} \cdot \overrightarrow{PQ} = 0$, $\overrightarrow{PQ} = \begin{pmatrix}0\\6\\2\end{pmatrix}$

b E.g. $\mathbf{r} = \begin{pmatrix}1\\-1\\3\end{pmatrix} + \lambda\begin{pmatrix}0\\6\\2\end{pmatrix}$

c $\begin{pmatrix}1\\2\\4\end{pmatrix}$

d 158°

6 a $\overrightarrow{AB} = \begin{pmatrix}6\\-2\\0\end{pmatrix}$

c $(36, 18, 0)$ d $5.10\ \mathrm{m\,s^{-1}}$

e 6 seconds f $(18, -6, 0)$