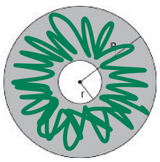
AB Calculus Volume of Solids of Revolution: Washer Method Name: ___________________________________

Geogebra Demo to go with this notesheet: https://goo.gl/RmTISe

In the last section, we learned how to find the volume of a solid that is generated by revolving a region around a line that is completely connected to the region using the disk method. But what happens if the region is not entirely connected to the axis of revolution? Well, let's start by looking at an example from geometry

Example 1 Find the area of the shaded region. Explain how you did it.



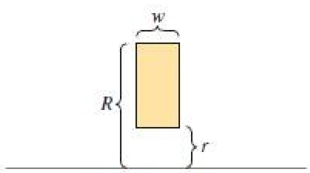
$A_{Big\,circle} = \pi R^2$

$A_{Little\,circle} = \pi r^2$

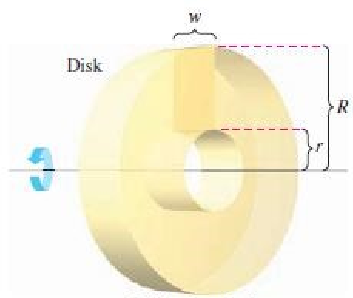
$\boxed{A_{gray} = \pi R^2 - \pi r^2}$

$\pi(R^2 - r^2)$

The disk method can be extended to cover solids of revolution with holes by replacing the representative disk with a representative **washer.** The washer is formed by revolving a rectangle about an axis, as shown below.



Axis of revolution



Solid of revolution

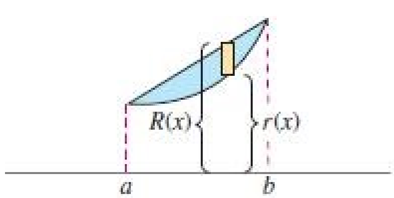
If r and R are the inner and outer radii of the washer and w is the width of the washer, the volume is given by

$$Volume = (\pi R^2 - \pi r^2)w = \pi(R^2 - r^2)w$$

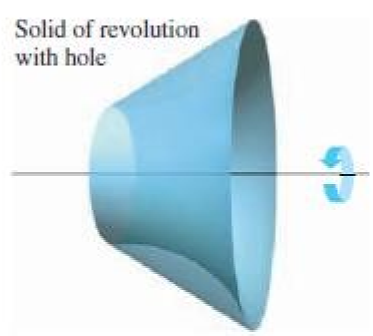
To see how this concept can be used to find the volume of a solid of revolution, consider a region bounded by an **outer radius** and an **inner radius** as shown in the image below. If the region is revolved about its axis of revolution, the volume of the resulting solid is given by

$$Volume = \pi \int_a^b ([R(x)]^2 - [r(x)]^2)dx$$

Note that the integral involving the inner radius represents the volume of the hole and is *subtracted* from the integral involving the outer radius.



Plane region



The Washer Method

To find the volume of a solid of revolution with the washer method, use one of the following formulas.

Horizontal Axis of Revolution	**Vertical Axis of Revolution**
$V = \pi \int_a^b \left([R(x)]^2 - [r(x)]^2\right) dx$	$V = \pi \int_a^b \left([R(y)]^2 - [r(y)]^2\right) dy$

Important: The rectangular strip is always perpendicular to the axis of revolution.

PERPENWASHULAR

Steps to Find the Volume of a Solid Using the Washer Method

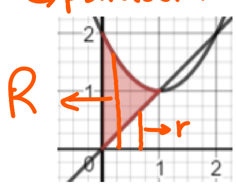
1. Draw a picture and shade the desired region.
2. Find R, the distance between the axis of revolution and the outside of the region.
3. Find r, the distance between the axis of revolution and the inside of the region.
4. Find the area of the base of one washer by substituting your expression for R and r into the formula for area, $A = \pi(R^2 - r^2)$.
5. Multiply by dx (if vertical strip) or dy (if horizontal strip) to get the volume of one slice of the solid.
6. Integrate over the interval to get the exact volume of the solid of revolution.

$(x-1)^2+1 = x \rightarrow x^2-2x+1+1 = x \rightarrow x^2-3x+2=0 \quad (x-2)(x-1) \quad x=1,2$ ↗ 45° Line

Example 2 Find the volume of the solid generated by rotating the region bounded by $y = x$, $x = 0$, and $y = (x-1)^2 + 1$ about the x-axis.

↳ y-axis

↳ parabola



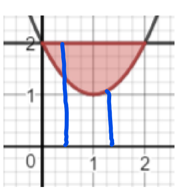
$R = (x-1)^2 + 1 - 0$

$r = x - 0$

$V = \pi \int_0^1 (R^2 - r^2)\,dx = \pi \int_0^1 \left(((x-1)^2+1)^2 - x^2\right) dx$

≈ 4.817

Example 3 Find the volume of the solid generated by rotating the region bounded by $y = 2$, and $y = x^2 - 2x + 2$ about the x-axis.



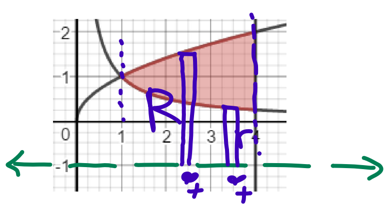
$R = 2 - 0$

$r = x^2 - 2x + 2 - 0$

$V = \pi \int_0^2 \left(2^2 - (x^2-2x+2)^2\right) dx \approx 13.404$

Example 4 Find the volume of the solid generated by rotating the region bounded by $y = \frac{1}{x}$, $y = \sqrt{x}$, and $x = 4$ about the line $y = -1$.

$R = \sqrt{x} - (-1)$

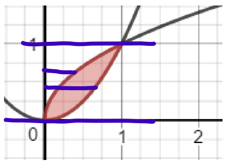


$r = \frac{1}{x} - (-1)$

$V = \pi \int_1^4 \left((\sqrt{x}+1)^2 - \left(\frac{1}{x}+1\right)^2\right) dx \approx 41.817$

Example 5 Find the volume of the solid generated by rotating the region bounded by $y = \sqrt{x}$ and $y = x^2$ about the y-axis.

$x = y^2 \quad x = \sqrt{y}$



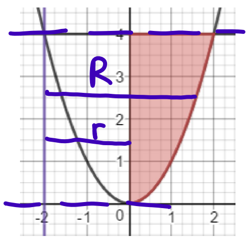
$R = \sqrt{y} - 0$

$r = y^2 - 0$

$V = \pi \int_0^1 (\sqrt{y})^2 - (y^2)^2 \, dy$

Example 6 Find the volume of the solid generated by rotating the region bounded by $y = 4$, $y = x^2$, and $x = 0$ about the line $x = -2$.

$\sqrt{y} = x$



$R = \sqrt{y} - -2$

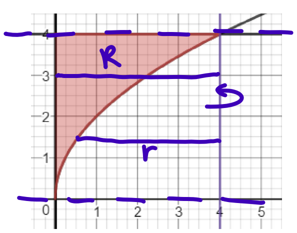
$r = 0 - -2$

$V = \pi \int_0^4 \left((\sqrt{y} + 2)^2 - (2)^2 \right) dy \approx 92\ 153$

Example 7 Find the volume of the solid generated by rotating the region bounded by $y = 2\sqrt{x}$, $y = 4$, and $x = 0$ about the line $x = 4$.

$y^2 = 4x$

$\frac{y^2}{4} = x$



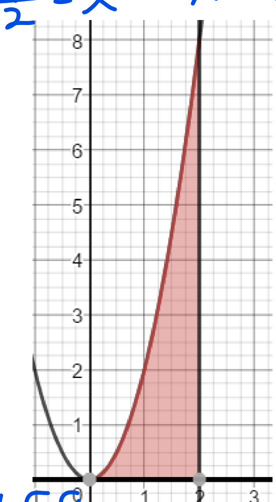
$R = 4 - 0 = 4$

$r = 4 - \frac{y^2}{4}$

$V = \pi \int_0^4 \left(4^2 - \left(4 - \frac{y^2}{4}\right)^2 \right) dy \approx 93\ 829$

Example 8 Let R be the region bounded by the graphs of $y = 2x^2$, $y = 0$, and $x = 2$.

$\frac{y}{2} = x^2 \quad x = \sqrt{\frac{y}{2}}$



a) Find the volume of the solid generated by revolving the region R about the x-axis.

$R = 2x^2 - 0$

$r =$ none (it's a disk)

$V = \pi \int_0^2 (2x^2)^2 \, dx \approx 80\ 425$

b) Find the volume of the solid generated by revolving the region R about the y-axis.

$R = 2 - 0$

$r = \sqrt{\frac{y}{2}} - 0$

$V = \pi \int_0^8 \left(2^2 - \left(\sqrt{\frac{y}{2}}\right)^2 \right) dy \approx 50\ 265$

c) Find the volume of the solid generated by revolving the region R about the line $y = 8$.

$R = 8 - 0 \quad r = 8 - x^2 \quad V = \pi \int_0^2 \left(8^2 - (8 - 2x^2)^2 \right) dx \approx 187\ 658$

d) Find the volume of the solid generated by revolving the region R about the line $x = 2$.

$R = 2 - \sqrt{\frac{y}{2}}$

$V = \pi \int_0^8 \left(2 - \sqrt{\frac{y}{2}} \right)^2 dy \approx 16.755$

no "r" (disk)